

## ON THE HYPERBOLICITY OF LORENZ RENORMALIZATION

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ABSTRACT. We consider infinitely renormalizable Lorenz maps with real critical exponent  $\alpha > 1$  and combinatorial type which is monotone and satisfies a long return condition. For these combinatorial types we prove the existence of periodic points of the renormalization operator, and that each map in the limit set of renormalization has an associated unstable manifold. An unstable manifold defines a family of Lorenz maps and we prove that each infinitely renormalizable combinatorial type (satisfying the above conditions) has a unique representative within such a family. We also prove that each infinitely renormalizable map has no wandering intervals and that the closure of the forward orbits of its critical values is a Cantor attractor of measure zero.

## 1. INTRODUCTION

Flows in three and higher dimensions can exhibit chaotic behavior and are far from being classified. Understanding higher dimensional flows is important since these have ties to physical systems, or at least simplifications thereof. The simplest example is that of the Lorenz equations. This three-dimensional flow is an approximate model for a convection flow in a box. In this paper we study *geometric* Lorenz flows since this class: (1) exhibits a wide range of dynamically complex behavior, (2) is “large” as a subset in the set of three-dimensional flows (in particular, it is open), and (3) is intimately connected with the Lorenz equations and as such has a physical significance.<sup>1</sup> We will describe the dynamics of individual infinitely renormalizable geometric Lorenz flows as well as the structure of the class of these infinitely renormalizable geometric Lorenz flows. The precise renormalization structure will be discussed later.

Recall that a *geometric Lorenz flow* is a flow whose associated vector field has a singularity of saddle type with a two-dimensional stable manifold  $\mathcal{W}^s$  and one-dimensional unstable manifold. The global dynamics of the flow should be such that there exists a two-dimensional transverse section  $S$  to the stable manifold which is divided into two components by the stable manifold and such that the first-return map  $F : S \setminus \mathcal{W}^s \rightarrow S$  is well-defined (points on  $\mathcal{W}^s$  can never return as they end up on the saddle point which is why  $F$  is undefined on  $\mathcal{W}^s$ ). The final condition is that  $S$  has a smooth  $F$ -invariant foliation whose leaves are exponentially contracted by  $F$ .

Under the above conditions  $F$  is a well-defined map on the leaves of the invariant foliation and by taking a quotient over leaves we get an interval map  $f : I \setminus \{c\} \rightarrow I$ ,

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<sup>1</sup>The reason why we consider geometric Lorenz flows instead of the Lorenz equations is that first-return maps of geometric flows automatically have nice properties, whereas for the Lorenz equations we would have to prove that such first-return maps exist, which is hard.

where  $I \subset \mathbb{R}$  and  $c$  corresponds to the stable manifold. Such a map is called a *Lorenz map*. Let  $L$  and  $R$  be the left and right components of  $I \setminus \{c\}$ , respectively. From the construction of  $f$  it follows that  $f|_L$  is equal to  $-|x|^\alpha$  in a left neighborhood of 0 up to a rescaling of the domain and range, and  $f|_R$  is equal to  $|x|^\alpha$  in a right neighborhood of 0, again up to a rescaling of the domain and range. The parameter  $\alpha > 0$  is the *critical exponent* which by construction equals the absolute value of the ratio between the weak stable eigenvalue and the unstable eigenvalue of the saddle point of the flow. Note in particular that there is no “preferred” value for  $\alpha$ , for example it is *not* an integer generically, it really is just an arbitrary (positive) real number. The expanding case  $\alpha < 1$  has been studied extensively elsewhere; we will consider the significantly harder case  $\alpha > 1$  where there is a delicate interplay between both expansion *and* contraction.

A Lorenz map is *renormalizable* if there exists an open interval around the critical point on which the first-return map is again a Lorenz map and the operator which takes a map to its first-return map is called a *renormalization operator*. The critical point divides the return interval into two halves, the forward orbits of which determine the *combinatorial type* of the renormalization. For example, the type (01, 100) encodes that the left half (01) is mapped to the right of the critical point (01) and then returns, whereas the right half (100) is first mapped to the left of the critical point (100) and then left again (100) before it returns. We will consider infinitely renormalizable maps (i.e. maps with a full forward orbit under the renormalization operator) of monotone type (i.e. types of the form (01  $\cdots$  1, 10  $\cdots$  0)) where the number of steps taken to return is much larger for one half of the return interval than it is for the other half (the precise condition can be found in Section 4). It is *very* difficult to deal with arbitrary combinatorial types, so we had to make some restrictions in order to make any progress.

The main part of this study relates to the hyperbolic properties of the renormalization operator and shows in particular that this operator has an expanding invariant cone-field on a renormalization invariant domain. This implies that each monotone Lorenz family (see Section 11) has a unique representative for every infinitely renormalizable combinatorial type, see Theorem 11.4. Contrast this with the important result of monotonicity of entropy for families of unimodal maps which essentially states that every nonperiodic kneading sequence is realized by a unique map in the family.

We also show that every point in the limit set of the renormalization operator has an associated unstable manifold and that the intersection of an unstable manifold and the set of infinitely renormalizable maps is a Cantor set, see Theorem 12.5. We believe the unstable manifolds to be two-dimensional, but are only able to show that their dimension is at least two.

Regarding the topological properties of the renormalization operator we show that there exists a periodic point of the renormalization operator for every periodic combinatorial type, see Theorem 6.1.

The main conclusion for the dynamics of an individual infinitely renormalizable Lorenz map is the absence of wandering intervals: two Lorenz maps of the same infinite renormalization type are topologically conjugated. We prove this result by showing that infinitely renormalizable maps satisfy the weak Markov property of [Martens, 1994] and hence cannot have a wandering interval, see Theorems 3.10 and 5.2. This is the first nonwandering interval result for Lorenz maps. The

nonexistence of wandering intervals for general Lorenz maps is still wide open and deserves attention.

We also prove that the closure of the orbits of the critical points of an infinitely renormalizable map is a Cantor attractor of zero Lebesgue measure, see Theorem 5.3.

Finally, let us briefly discuss the techniques employed in the proofs. The general idea is that by making one return time large we get a first-return map which is essentially  $|x|^\alpha$  up to scaling by maps which are close to being affine. This allows us to explicitly calculate an almost invariant set of the renormalization operator. This is done in Section 4 which is the first major part of this paper. After this hurdle we are able to prove properties of individual infinitely renormalizable maps (no wandering intervals, Cantor attractor, periodic points of renormalization) in Sections 5 and 6.

The second major part is calculating the derivative of the renormalization operator on a neighborhood of the limit set of renormalization. This is done in Section 9. However, calculating the derivative of the renormalization operator defined on interval maps is rather hopeless so we need a better representation of the domain of the renormalization operator. The representation we choose are the so-called *decompositions* which are families of diffeomorphisms parametrized by ordered countable sets (see Section 7). The renormalization operator is semi-conjugate to an operator on (essentially) a space of decompositions. There are two main reasons why the derivative of this operator is easier to compute: (1) the limit set is essentially a Hilbert cube (see Proposition 8.8), and (2) deformations in any of the countably many directions are monotone in a sense explained in Section 9. The first point means that the derivative is just an infinite matrix and the second allows us to calculate just a few partial derivatives and then make sweeping estimates for the remaining (countably infinite) directions.

After having computed the derivative we are able to construct an invariant cone field in Section 10. A by-product of the derivative calculations is that the derivative is orientation-preserving in the unstable direction and using this together with the invariant cone field we are able to prove the association of each combinatorial type with a unique representative of a monotone family of maps in Section 11. The invariant cone-field also implies the existence of unstable manifolds in the limit set of renormalization, see Section 12.

As a closing remark we point out that in order to deal with arbitrary critical exponents  $\alpha > 1$  we had to invent real analytic methods. To our knowledge, this work is the first to analyze the hyperbolic structure of the limit of renormalization for arbitrary critical exponents.

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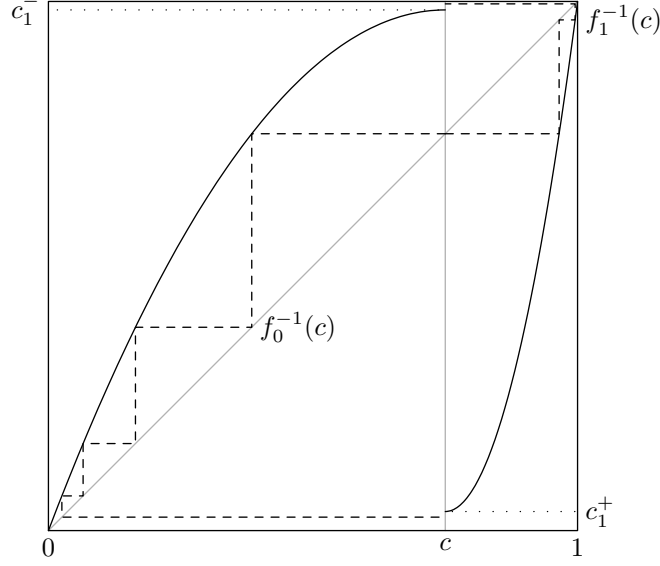


FIGURE 1. Illustration of the graph of a (01, 1000)-renormalizable Lorenz map.

## 2. THE RENORMALIZATION OPERATOR

In this section we define the renormalization operator on Lorenz maps and introduce notation that will be used throughout.

**Definition 2.1.** The standard Lorenz family  $(u, v, c) \mapsto Q(x)$  is defined by

$$(1) \quad Q(x) = \begin{cases} u \cdot \left(1 - \left(\frac{c-x}{c}\right)^\alpha\right), & \text{if } x \in [0, c), \\ 1 + v \cdot \left(-1 + \left(\frac{x-c}{1-c}\right)^\alpha\right), & \text{if } x \in (c, 1], \end{cases}$$

where  $u \in [0, 1]$ ,  $v \in [0, 1]$ ,  $c \in (0, 1)$ , and  $\alpha > 1$ . The parameter  $\alpha$  is called the critical exponent and will be fixed once and for all.

*Remark 2.2.* The parameters  $(u, v, c)$  are chosen so that: (i)  $u$  is the length of the image of  $[0, c)$ , (ii)  $v$  is the length of the image of  $(c, 1]$ , (iii)  $c$  is the critical point (which is the same as the point of discontinuity). Note that  $u$  and  $1 - v$  are the critical values of  $Q$ .

**Definition 2.3.** A  $\mathcal{C}^k$ -Lorenz map  $f$  on  $[0, 1] \setminus \{c\}$  is any map which can be written as

$$(2) \quad f(x) = \begin{cases} \phi \circ Q(x), & \text{if } x \in [0, c), \\ \psi \circ Q(x), & \text{if } x \in (c, 1], \end{cases}$$

where  $\phi, \psi \in \mathcal{D}^k$  are orientation-preserving  $\mathcal{C}^k$ -diffeomorphisms on  $[0, 1]$ , called the diffeomorphic parts of  $f$ . See Figure 1 for an illustration of a Lorenz map. The set of  $\mathcal{C}^k$ -Lorenz maps is denoted  $\mathcal{L}^k$ ; the subset  $\mathcal{L}^S \subset \mathcal{L}^3$  denotes the Lorenz maps with negative Schwarzian derivative (see Appendix C for more information on the Schwarzian derivative).

A Lorenz map has two critical values which we denote

$$c_1^- = \lim_{x \uparrow c} f(x) \quad \text{and} \quad c_1^+ = \lim_{x \downarrow c} f(x).$$

If  $c_1^+ < c < c_1^-$  then  $f$  is nontrivial, otherwise all points converge to some fixed point under iteration and for this reason  $f$  is called trivial. Unless otherwise noted, we will always assume all maps to be nontrivial.

We make the identification

$$\mathcal{L}^k = [0, 1]^2 \times (0, 1) \times \mathcal{D}^k \times \mathcal{D}^k,$$

by sending  $(u, v, c, \phi, \psi)$  to  $f$  defined by (2). Note that  $(u, v, c)$  defines  $Q$  in (2) according to (1). For  $k \geq 2$  this identification turns  $\mathcal{L}^k$  into a subset of the Banach space  $\mathbb{R}^3 \times \mathcal{D}^k \times \mathcal{D}^k$ . Here  $\mathcal{D}^k$  is endowed with the Banach space structure of  $\mathcal{C}^{k-2}$  via the nonlinearity operator. In particular, this turns  $\mathcal{L}^k$  into a metric space. For  $k < 2$  we turn  $\mathcal{L}^k$  into a metric space by using the usual  $\mathcal{C}^k$  metric on  $\mathcal{D}^k$ . See Appendix B for more information on the Banach space  $\mathcal{D}^k$ .

*Remark 2.4.* It may be worth emphasizing that for  $k \geq 2$  we are *not* using the linear structure induced from  $\mathcal{C}^k$  on the diffeomorphisms  $\mathcal{D}^k$ . Explicitly, if  $\phi, \psi \in \mathcal{D}^k$  and  $N$  denotes the nonlinearity operator, then

$$a\phi + b\psi = N^{-1}(aN\phi + bN\psi), \quad \forall a, b \in \mathbb{R},$$

and

$$\|\phi\|_{\mathcal{D}^k} = \|N\phi\|_{\mathcal{C}^{k-2}}.$$

We call this norm on  $\mathcal{D}^k$  the  $\mathcal{C}^{k-2}$ -nonlinearity norm. The nonlinearity operator  $N : \mathcal{D}^k \rightarrow \mathcal{C}^{k-2}$  is a bijection and is defined by

$$N\phi(x) = D \log D\phi(x).$$

See Appendix B for more details on the nonlinearity operator.

We now define the renormalization operator for Lorenz maps.

**Definition 2.5.** A Lorenz map  $f$  is renormalizable if there exists an interval  $C \subsetneq [0, 1]$  (properly containing  $c$ ) such that the first-return map to  $C$  is affinely conjugate to a nontrivial Lorenz map. Choose  $C$  so that it is maximal with respect to these properties. The first-return map affinely rescaled to  $[0, 1]$  is called the renormalization of  $f$  and is denoted  $\mathcal{R}f$ . The operator  $\mathcal{R}$  which sends  $f$  to its renormalization is called the renormalization operator.

Explicitly, if  $f$  is renormalizable then there exist minimal positive integers  $a$  and  $b$  such that the first return map  $\tilde{f}$  to  $C$  is given by

$$\tilde{f}(x) = \begin{cases} f^{a+1}(x), & \text{if } x \in L, \\ f^{b+1}(x), & \text{if } x \in R, \end{cases}$$

where  $L$  and  $R$  are the left and right components of  $C \setminus \{c\}$ , respectively. The renormalization of  $f$  is defined by

$$\mathcal{R}f(x) = h^{-1} \circ \tilde{f} \circ h(x), \quad x \in [0, 1] \setminus \{h^{-1}(c)\},$$

where  $h : [0, 1] \rightarrow C$  is the affine orientation-preserving map taking  $[0, 1]$  to  $C$ . Note that  $C$  is chosen maximal so that  $\mathcal{R}f$  is uniquely defined.

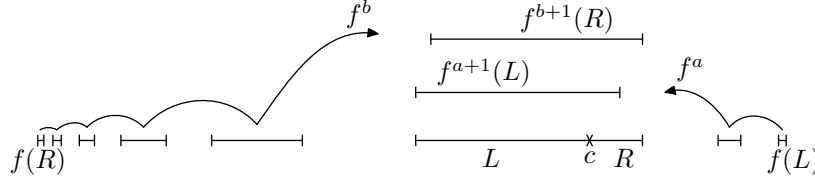


FIGURE 2. Illustration of the dynamical intervals of a Lorenz map which is  $\omega$ -renormalizable, with  $\omega = (011, 100000)$ ,  $a = 2$ ,  $b = 5$ .

*Remark 2.6.* We would like to emphasize that the renormalization is assumed to be a nontrivial Lorenz map. It is possible to define the renormalization operator for maps whose renormalization is trivial but we choose not to include these in our definition. Such maps can be thought of as degenerate and including them makes some arguments more difficult which is why we choose to exclude them.

Next, we wish to describe the combinatorial information encoded in a renormalizable map.

**Definition 2.7.** A branch of  $f^n$  is a maximal open interval  $B$  on which  $f^n$  is monotone (here maximality means that if  $A$  is an open interval which properly contains  $B$ , then  $f^n$  is not monotone on  $A$ ).

To each branch  $B$  of  $f^n$  we associate a word  $w(B) = \sigma_0 \cdots \sigma_{n-1}$  on symbols  $\{0, 1\}$  by

$$\sigma_j = \begin{cases} 0 & \text{if } f^j(B) \subset (0, c), \\ 1 & \text{if } f^j(B) \subset (c, 1), \end{cases}$$

for  $j = 0, \dots, n-1$ .

**Definition 2.8.** Assume  $f$  is renormalizable and let  $a, b, L$  and  $R$  be as in Definition 2.5. The forward orbits of  $L$  and  $R$  induce a pair of words  $\omega = (w(\hat{L}), w(\hat{R}))$  called the type of renormalization, where  $\hat{L}$  is the branch of  $f^{a+1}$  containing  $L$  and  $\hat{R}$  is the branch of  $f^{b+1}$  containing  $R$ . In this situation we say that  $f$  is  $\omega$ -renormalizable. See Figure 2 for an illustration of these definitions.

Let  $\bar{\omega} = (\omega_0, \omega_1, \dots)$ . If  $\mathcal{R}^n f$  is  $\omega_n$ -renormalizable for  $n = 0, 1, \dots$ , then we say that  $f$  is infinitely renormalizable and that  $f$  has combinatorial type  $\bar{\omega}$ . If the length of both words of  $\omega_k$  is uniformly bounded in  $k$ , then  $f$  is said to have bounded combinatorial type.

The set of  $\omega$ -renormalizable Lorenz maps is denoted  $\mathcal{L}_\omega$ . We will use variations of this notation as well; for  $\bar{\omega} = (\omega_0, \dots, \omega_{n-1})$  we let  $\mathcal{L}_{\bar{\omega}}$  denote the set of Lorenz maps  $f$  such that  $\mathcal{R}^i f$  is  $\omega_i$ -renormalizable, for  $i = 0, \dots, n-1$ , and similarly if  $n = \infty$ . Furthermore, if  $\Omega$  is a set of types of renormalization, then  $\mathcal{L}_\Omega$  denotes the set of Lorenz maps which are  $\omega$ -renormalizable for some  $\omega \in \Omega$ .

We will almost exclusively restrict our attention to monotone combinatorics, that is renormalizations of type

$$\omega = (\overbrace{01 \cdots 1}^a, \overbrace{10 \cdots 0}^b).$$

In what follows we will need to know how the five-tuple representation of a Lorenz map changes under renormalization. It is not difficult to write down the

formula for any type of renormalization but it becomes a bit messy so we restrict ourselves to monotone combinatorics. However, first we need to introduce the zoom operator.

**Definition 2.9.** The zoom operator  $Z$  takes a diffeomorphism and rescales it affinely to a diffeomorphism on  $[0, 1]$ . Explicitly, let  $g$  be a map and  $I$  an interval such that  $g|_I$  is an orientation-preserving diffeomorphism. Define

$$Z(g; I) = \zeta_{g(I)}^{-1} \circ g \circ \zeta_I,$$

where  $\zeta_A : [0, 1] \rightarrow A$  is the orientation-preserving affine map which takes  $[0, 1]$  onto  $A$ . See Appendix B for more information on zoom operators.

*Remark 2.10.* The terminology “zoom operator” is taken from Martens [1998], but our definition is somewhat simpler since we only deal with orientation-preserving diffeomorphisms. We will use the words ‘rescale’ and ‘zoom’ synonymously.

**Lemma 2.11.** *If  $f = (u, v, c, \phi, \psi)$  is renormalizable of monotone combinatorics, then*

$$\mathcal{R}f = (u', v', c', \phi', \psi')$$

*is given by*

$$\begin{aligned} u' &= \frac{|Q(L)|}{|U|}, & v' &= \frac{|Q(R)|}{|V|}, & c' &= \frac{|L|}{|C|}, \\ \phi' &= Z(f_1^a \circ \phi; U), & \psi' &= Z(f_0^b \circ \psi; V), \end{aligned}$$

where  $U = \phi^{-1} \circ f_1^{-a}(C)$  and  $V = \psi^{-1} \circ f_0^{-b}(C)$ .

*Proof.* This follows from two properties of zoom operators: (i) the map  $q(x) = x^\alpha$  on  $[0, 1]$  is ‘fixed’ under zooming on intervals adjacent to the critical point, that is  $Z(q; (0, t)) = q$  for  $t \in (0, 1)$  (technically speaking we have not defined  $Z$  in this situation, but applying the formula for  $Z$  will give this result), and (ii) zoom operators satisfy  $Z(h \circ g; I) = Z(h; g(I)) \circ Z(g; I)$ .  $\square$

**Notation.** The notation introduced in this section will be used repeatedly throughout. Here is a quick summary.

A Lorenz map is denoted either  $f$  or  $(u, v, c, \phi, \psi)$  and these two notations are used interchangeably. Sometimes we write  $f_0$  or  $f_1$  to specify that we are talking about the left or right branch of  $f$ , respectively. Similarly, when talking about the inverse branches of  $f$ , we write  $f_0^{-1}$  and  $f_1^{-1}$ . The subscript notation is also used for the standard family  $Q$  (so  $Q_0$  denotes the left branch, etc.).

A Lorenz map has one critical point  $c$  and two critical values which we denote  $c_1^- = \lim_{x \uparrow c} f(x)$  and  $c_1^+ = \lim_{x \downarrow c} f(x)$ . The critical exponent is denoted  $\alpha$  and is always assumed to be fixed to some  $\alpha > 1$ .

In general we use primes for variables associated with the renormalization of  $f$ . For example  $(u', v', c', \phi', \psi') = \mathcal{R}f$ . Sometimes we use parentheses instead of primes, for example  $c_1^-(\mathcal{R}f)$  denotes the left critical value of  $\mathcal{R}f$ . In order to avoid confusion, we try to use  $D$  consistently to denote derivative instead of using primes.

With a renormalizable  $f$  we associate a return interval  $C$  such that  $C \setminus \{c\}$  has two components which we denote  $L$  and  $R$ . We use the notation  $a + 1$  and  $b + 1$  to denote the return times of the first-return map to  $C$  from  $L$  and  $R$ , respectively. The letters  $U$  and  $V$  are reserved to denote the pull-backs of  $C$  as in Lemma 2.11. We let  $U_1 = \phi(U)$ ,  $U_{i+1} = f^i(U_1)$  for  $i = 1, \dots, a$ , and  $V_1 = \psi(V)$ ,  $V_{j+1} = f^j(V_1)$

for  $j = 1, \dots, b$  (note that  $U_{a+1} = C = V_{b+1}$ ). We call  $\{U_i\}$  and  $\{V_j\}$  the cycles of renormalization.

### 3. GENERALIZED RENORMALIZATION

In this section we adapt the idea of generalized renormalization introduced by Martens [1994]. The central concept is the weak Markov property which is related to the distortion of the monotone branches of iterates of a map.

**Definition 3.1.** An interval  $C$  is called a nice interval of  $f$  if: (i)  $C$  is open, (ii) the critical point of  $f$  is contained in  $C$ , and (iii) the orbit of the boundary of  $C$  is disjoint from  $C$ .

*Remark 3.2.* A ‘nice interval’ is analogous to a ‘nice point’ for unimodal maps [see Martens, 1994]. The difference is that for unimodal maps one point suffices to define an interval around the critical point (the ‘other’ boundary point is a preimage of the first), whereas for Lorenz maps the boundary points of a nice interval are independent. The term ‘nice’ is perhaps a bit vague but its use has become established by now.

**Definition 3.3.** Fix  $f$  and a nice interval  $C$ . The transfer map to  $C$  induced by  $f$ ,

$$T : \bigcup_{n \geq 0} f^{-n}(C) \rightarrow C,$$

is defined by  $T(x) = f^{\tau(x)}(x)$ , where

$$\tau : \bigcup_{n \geq 0} f^{-n}(C) \rightarrow \mathbb{N}$$

is the transfer time to  $C$ ; that is  $\tau(x)$  is the smallest nonnegative integer  $n$  such that  $f^n(x) \in C$ .

*Remark 3.4.* Note that: (i) the domain of  $T$  is open, since  $C$  is open by assumption, and  $f^{-1}(U)$  is open if  $U$  is open (even if  $U$  contains a critical value), since the point of discontinuity of  $f$  is not in the domain of  $f$ , (ii)  $T$  is defined on  $C$  and  $T|_C$  equals the identity map on  $C$ .

**Proposition 3.5.** Let  $T$  be the transfer map of  $f$  to a nice interval  $C$ . If  $I$  is a component of the domain of  $T$ , then  $\tau|_I$  is constant and  $I$  is mapped monotonically onto  $C$  by  $f^{\tau(I)}$ . Furthermore  $I, f(I), \dots, f^{\tau(I)}(I)$  are pairwise disjoint.

*Remark 3.6.* This means in particular that the components of the domain of  $T$  are the same as the branches of  $T$ . In what follows we will use the terminology “a branch of  $T$ ” interchangeably with “a component of the domain of  $T$ ”.

*Proof.* If  $I = C$  then the proposition is trivial since  $T|_C$  is the identity map on  $C$ , so assume that  $I \neq C$ .

Pick some  $x \in I$  and let  $n = \tau(x)$ . Note that  $n > 0$  since  $I \neq C$ . We claim that the branch  $B$  of  $f^n$  containing  $x$  is mapped over  $C$ . From this it immediately follows that  $\tau|_I = n$  and  $f^n(I) = C$ .

Since  $f^n|_B$  is monotone and  $f(x) \in C$  it suffices to show that  $f^n(\partial B) \cap C = \emptyset$ . To this end, let  $y \in \partial B$ . Then there exists  $0 \leq i < n$  such that  $f^i(y) \in \{0, c, 1\}$ .

If  $f^i(y) \in \{0, 1\}$  then we are done, since these points are fixed by  $f$ .

So assume that  $f^i(y) = c$  and let  $J = (x, y)$ . Then  $f^i(J) \cap \partial C \neq \emptyset$  since  $f^i(x) \notin C$  by minimality of  $\tau(x)$ . Consequently  $f^n(y) \notin C$ , otherwise  $f^n(J) \subset C$



which would imply  $f^{n-i}(\partial C) \cap C \neq \emptyset$ . But this is impossible since  $C$  is nice and hence the claim follows.

From  $\tau(I) = n$  it follows that  $I, \dots, f^n(I)$  are pairwise disjoint. Suppose not, then  $J = f^i(I) \cap f^j(I)$  is nonempty for some  $0 \leq i < j \leq n$ . But then the transfer time on  $I \cap f^{-i}(J)$  is at most  $i + (n - j)$  which is strictly smaller than  $n$ , and this contradicts the fact that  $\tau(I) = n$ .  $\square$

**Proposition 3.7.** *Assume that  $f$  has no periodic attractors and that  $Sf < 0$ . Let  $T$  be the transfer map of  $f$  to a nice interval  $C$ . Then the complement of the domain of  $T$  is a compact,  $f$ -invariant and hyperbolic set (and consequently it has zero Lebesgue measure).*

*Proof.* Let  $U = \text{dom } T$  and let  $\Gamma = [0, 1] \setminus U$ .

Since  $U$  is open  $\Gamma$  is closed and hence compact (since it is obviously bounded).

By definition  $f^{-1}(U) \subset U$  which implies  $f(\Gamma) \subset \Gamma$ .

We can characterize  $\Gamma$  as the set of points  $x$  such that  $f^n(x) \notin C$  for all  $n \geq 0$ . Since  $Sf < 0$  it follows that  $f$  cannot have nonhyperbolic periodic points in  $\Gamma$  [Misiurewicz, 1981, Theorem 1.3] and by assumption  $f$  has no periodic attractors so  $\Gamma$  must be hyperbolic [de Melo and van Strien, 1993, Theorem III.3.2].<sup>2</sup>

Finally, it is well known that a compact, invariant and hyperbolic set has zero Lebesgue measure if  $f$  is at least  $\mathcal{C}^{1+\text{H\"older}}$  [de Melo and van Strien, 1993, Theorem III.2.6].<sup>2</sup>  $\square$

**Definition 3.8.** A map  $f$  is said to satisfy the weak Markov property if there exists a  $\delta > 0$  and a nested sequence of nice intervals  $C_1 \supset C_2 \supset \dots$ , such that  $C_n$  contains a  $\delta$ -scaled neighborhood of  $C_{n+1}$  and such that the transfer map to  $C_n$  is defined almost everywhere, for every  $n > 0$ .

*Remark 3.9.* If  $I \subset J$  are two intervals, then  $J$  is said to contain a  $\delta$ -scaled neighborhood of  $I$  if  $J \setminus I$  consists of two components  $I_0$  and  $I_1$ , and if  $|I_k| > \delta|I|$  for  $k = 0, 1$ .

The relevance of this property in Definition 3.8 is that it can be used in conjunction with the Koebe lemma to control the distortion of the transfer map to  $C_n$ .

**Theorem 3.10.** *If  $f$  satisfies the weak Markov property, then  $f$  has no wandering intervals.*

*Proof.* In order to reach a contradiction assume that there exists a wandering interval  $W$  which is not contained in a strictly larger wandering interval.

Note that the orbit of  $W$  must accumulate on at least one side of  $c$ . Otherwise there would exist an interval  $I$  disjoint from the orbit of  $W$  with  $c \in \text{cl } I$ . We could then modify  $f$  on  $I$  in such a way that the resulting map would be a bimodal  $\mathcal{C}^2$ -map with nonflat critical points and  $W$  would still be a wandering interval for the modified map, see Figure 3. However, such maps do not have wandering intervals [Martens et al., 1992].

Now let  $\{C_k\}$  be the sequence of nice intervals that we get from the weak Markov property and let  $T_k$  denote the transfer map to  $C_k$ . We claim that  $W \subset \text{dom } T_k$ . To see this, note that  $f^{n_k}(W) \cap C_k \neq \emptyset$  for some minimal  $n_k$ , since the orbit of  $W$

<sup>2</sup> The theorems from de Melo and van Strien [1993] that are referenced in this proof are stated for maps whose domain is an interval but their proofs go through, mutatis mutandis, for Lorenz maps.

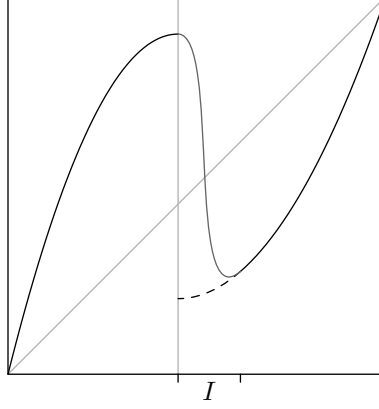


FIGURE 3. Illustration showing why the orbit of a wandering interval must accumulate on the critical point. If  $f$  has a wandering interval whose orbit does not intersect some (one-sided) neighborhood  $I$  of the critical point, then by modifying  $f$  on  $I$  according to the gray curve we create a bimodal map with a wandering interval. This is impossible since bimodal maps with nonflat critical points do not have wandering intervals.

accumulates on the critical point. But  $C_k$  is a nice interval, so in fact we must have  $f^{n_k}(W) \subset C_k$ , else there would exist  $x \in W$  such that  $f^{n_k}(x) \in \partial C_k$  and hence the orbit of  $x$  would never enter  $C_k$  which is impossible since  $W$  is wandering and its orbit accumulates on the critical point. This shows that  $W$  is contained in the domain of the transfer map to  $C_k$  as claimed.

Let  $B_k$  be the component of  $\text{dom } T_k$  which contains  $W$ . By Proposition 3.5  $T_k(B_k) = C_k$ . From the weak Markov property we get a  $\delta$  (not depending on  $k$ ) such that  $C_k$  contains a  $\delta$ -scaled neighborhood of  $C_{k+1}$ . Applying the Macroscopic Koebe lemma we can pull this space back to get that  $B_k$  contains a  $\delta'$ -scaled neighborhood of  $B_{k+1}$ , where  $\delta'$  only depends on  $\delta$ .

Now let  $B = \bigcap B_k$ . By the above  $B_k$  contains a  $\delta'$ -scaled neighborhood of  $W$  for every  $k$ , hence  $B$  strictly contains  $W$ . By Proposition 3.5 the collection  $\{f^i(B_k)\}_{i=0}^{n_k}$  is pairwise disjoint for every  $k$ . Thus  $B$  is a wandering interval which strictly contains the wandering interval  $W$  (note that  $n_k \rightarrow \infty$  since  $|C_k| \rightarrow 0$ ). This contradicts the maximality of  $W$  and hence  $f$  cannot have wandering intervals.  $\square$

**Theorem 3.11.** *If  $f$  satisfies the weak Markov property, then  $f$  is ergodic.*

*Proof.* In order to reach a contradiction, assume that there exist two invariant sets  $X$  and  $Y$  such that  $|X| > 0$ ,  $|Y| > 0$  and  $|X \cap Y| = 0$ . Let  $\{C_k\}$  be the sequence of nice intervals that we get from the weak Markov property. We claim that

$$\frac{|X \cap C_k|}{|C_k|} \rightarrow 1 \quad \text{and} \quad \frac{|Y \cap C_k|}{|C_k|} \rightarrow 1, \quad \text{as } k \rightarrow \infty.$$

Thus we arrive at a contradiction since this shows that  $|X \cap Y| > 0$ .

Let  $\Gamma_k$  be the complement of the domain of the transfer map to  $C_k$ . By the weak Markov property  $|\Gamma_k| = 0$ , hence  $\bigcup \Gamma_k$  also has zero measure. This and the

assumption that  $|X| > 0$  implies that there exists a density point  $x$  which lies in  $X$  as well as in the domain of the transfer map to  $C_k$ , for every  $k$ .

Let  $B_k$  be the branch of the transfer map to  $C_k$  containing  $x$ , and let  $\tau_k$  be the transfer time for  $B_k$ . We contend that  $|B_k| \rightarrow 0$ . If not, there would exist a subsequence  $\{k_i\}$  such that  $B = \bigcap B_{k_i}$  had positive measure, and thus  $B$  would be contained in a wandering interval (which is impossible by Theorem 3.10). Here we have used that  $C_k$  is a nice interval so the orbit of  $B_k$  satisfies the disjointness property of Proposition 3.5.

Since  $f^{\tau_k}(B_k) = C_k$  we can use the weak Markov property and the Koebe lemma to get that there exists  $K < \infty$  (not depending on  $k$ ) such that the distortion of  $f^{\tau_k}$  on  $B_k$  is bounded by  $K$ . This, together with the assumption that  $f(X) \subset X$ , shows that

$$\frac{|C_k \setminus X|}{|C_k|} \leq \frac{|f^{\tau_k}(B_k \setminus X)|}{|f^{\tau_k}(B_k)|} \leq K \frac{|B_k \setminus X|}{|B_k|} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

The last step follows from  $x$  being a density point, since  $|B_k| \rightarrow 0$ .

Now apply the same argument to  $Y$  and the claim follows.  $\square$

#### 4. THE INVARIANT SET

In this section we construct an ‘invariant’ and relatively compact set for the renormalization operator. This construction works for types of renormalization where the return time of one branch is much longer than the other. This result will be exploited in the following sections.

**Definition 4.1.** Fix  $\alpha > 1$ ,  $\sigma \in (0, 1)$ ,  $\beta \in (0, (\sigma/\alpha)^2)$  and let  $b_0 \in \mathbb{N}$  be a free parameter. Define  $\Omega$  to be the following (finite) set of monotone types

$$(3) \quad \Omega = \left\{ (0 \overbrace{1 \cdots 1}^a, 1 \overbrace{0 \cdots 0}^b) \mid \alpha + \sigma \leq a + 1 \leq 2\alpha - \sigma, \quad b_0 \leq b \leq (1 + (\sigma/\alpha)^2 - \beta)b_0 \right\}.$$

(Note that  $a$  and  $b$  are integers.) We assume that  $\sigma$  has been chosen so that the two inequalities involving  $a$  have at least one integer solution.<sup>3</sup>

Let  $\delta = (1/b_0)^2$ ,  $\varepsilon = 1 - c$  (note that  $\varepsilon$  depends on  $f$ ) and define

$$(4) \quad \mathcal{K} = \{f \in \mathcal{L}^1 \mid \alpha^{-b_0/\alpha} \leq \varepsilon \leq \theta \alpha^{-b_0\sigma/\alpha^2}, \text{ Dist } \phi \leq \delta, \text{ Dist } \psi \leq \delta\},$$

where  $\theta > 1$  is a constant not depending on  $b_0$ .<sup>4</sup> We assume that  $b_0$  is large enough for the two inequalities involving  $\varepsilon$  to have at least one solution.

We are going to show that  $\mathcal{K}$  is ‘invariant’ under the restriction of  $\mathcal{R}$  to types in  $\Omega$  as long as  $b_0$  is large enough. Recall that  $f \in \mathcal{L}_\Omega^S$  if and only if  $f$  has negative Schwarzian derivative and is  $\omega$ -renormalizable for some  $\omega \in \Omega$ .

**Theorem 4.2.** *If  $f \in \mathcal{L}_\Omega^S$  and  $1 - c_1^+(\mathcal{R}f) \geq \lambda > 0$  for some constant  $\lambda$  (not depending on  $b_0$ ), then*

$$f \in \mathcal{K} \implies \mathcal{R}f \in \mathcal{K},$$

for  $b_0$  large enough.

<sup>3</sup>For  $\alpha \in (1, 2]$  there is exactly one integer solution if  $\sigma$  is small enough. For  $\alpha > 2$  it is possible to choose  $\sigma$  so that there are at least two solutions.

<sup>4</sup>The constant  $\theta$  is given by Proposition 4.10.

The condition on  $c_1^+(\mathcal{R}f)$  is a bit unpleasant but we need it to exclude maps such that  $\varepsilon(\mathcal{R}f)$  is too small for us to deal with. This situation occurs when the right branch of the renormalization is trivial. We can work around this problem by considering twice renormalizable maps because such maps will automatically satisfy the condition on  $c_1^+(\mathcal{R}f)$ , and this leads us to:

**Theorem 4.3.** *If both  $f \in \mathcal{L}_\Omega^S$  and  $\mathcal{R}f \in \mathcal{L}_\Omega^S$ , then*

$$f \in \mathcal{K} \implies \mathcal{R}f \in \mathcal{K}.$$

for  $b_0$  large enough.

The proofs of Theorems 4.2 and 4.3 can be found at the end of this section.

*Remark 4.4.* The full family theorem [Martens and de Melo, 2001] implies that: (i) for every  $\lambda \in (0, 1)$  there exists  $f \in \mathcal{L}_\Omega^S \cap \mathcal{K}$  such that  $c_1^+(\mathcal{R}f) \geq 1 - \lambda$  (e.g. any  $f \in \mathcal{K}$  can be deformed in the  $(u, v)$  directions in such a way that  $f$  is renormalizable to a map such that  $c_1^+(\mathcal{R}f) = 0$ ), and (ii)  $\mathcal{K}$  intersects the set of twice renormalizable maps (of *any* combinatorics). This shows that both theorems above are not vacuous.

The main reason for introducing the set  $\mathcal{K}$  is the following:

**Proposition 4.5.**  *$\mathcal{K}$  is relatively compact in  $\mathcal{L}^0$ .*

*Proof.* Clearly  $\varepsilon(f)$  for  $f \in \mathcal{K}$  lies inside a compact set in  $(0, 1)$ . Hence we only need to show that the ball  $B = \{\phi \in \mathcal{D}^1([0, 1]) \mid \text{Dist } \phi \leq \delta\}$  is relatively compact in  $\mathcal{D}^0([0, 1])$ . This is an application of the Arzelà–Ascoli theorem; if  $\{\phi_n \in B\}$  then  $|\phi_n(y) - \phi_n(x)| \leq e^\delta |y - x|$  hence this sequence is equicontinuous (as well as uniformly bounded), so it has a uniformly convergent subsequence.  $\square$

The rest of this section is devoted to the proof of Theorem 4.2. We will need the following expressions for the inverse branches of  $f$  which can be derived from equations (1) and (2):

$$(5) \quad f_0^{-1}(x) = c - c \left( \frac{|\phi^{-1}([x, c_1^-])|}{|\phi^{-1}([0, c_1^-])|} \right)^{1/\alpha},$$

$$(6) \quad f_1^{-1}(x) = c + (1 - c) \left( 1 - \frac{|\psi^{-1}([x, 1])|}{|\psi^{-1}([c_1^+, 1])|} \right)^{1/\alpha}.$$

The following lemma gives us control over certain backward orbits of the critical point. This is later used to control the critical values and the derivative of the first-return map. The underlying idea for these results is that the backward orbit of  $c$  under  $f_0$  initially behaves like a root and eventually like a linear map whose multiplier is determined by  $Df(0)$ , whereas the backward orbit of  $c$  under  $f_1$  behaves like a linear map whose multiplier is determined by  $Df(1)$ .

**Lemma 4.6.** *There exist  $\mu > 0$  and  $\nu, k \in (0, 1)$  such that if  $f \in \mathcal{L}_\Omega^1$ ,  $\text{Dist } \phi \leq \delta$ ,  $\text{Dist } \psi \leq \delta$ ,  $\varepsilon \leq k$ , and  $\varepsilon \geq \gamma^{\alpha_{b_0}}$  for some  $\gamma \in (0, 1)$  not depending on  $b_0$ , then*

$$\frac{f_1^{-1}(c) - c}{\varepsilon} \geq 1 - \mu\varepsilon \quad \text{and} \quad \frac{c - f_0^{-n}(c)}{c} \geq \nu\varepsilon^{\alpha^{-n}},$$

for  $b_0$  large enough.

*Proof.* We claim that

$$(7) \quad f_1^{-1}(c) - c \geq \varepsilon \cdot \left(1 - \frac{e^\delta \varepsilon}{c - f_0^{-b}(c)}\right)^{1/\alpha},$$

$$(8) \quad c - f_0^{-n}(c) \geq c e^{-\delta/(\alpha-1)} (f_1^{-1}(c) - c)^{\alpha^{-n}}.$$

Assume for the moment that these equations hold. The idea of the proof is that if we have some initial lower bound on  $f_1^{-1}(c) - c$  then we can plug that into (8) (with  $n = b$ ), and this bound in turn can be plugged back into (8) to get a new lower bound on  $f_1^{-1}(c) - c$ . We will show that iterating the initial bound in this way will actually improve it and that this iterative procedure will lead to the desired statement. Finally we show that there exists an initial bound that is good enough to start off the iteration.

To begin with consider (7). This equation follows from a computation using (6) and the fact that  $1 - c_1^+ > c - f_0^{-b}(c)$  holds for monotone combinatorics.

Next, we prove (8). Apply (5) to get

$$f_0^{-1}(x) \leq c - c \left( e^{-\delta} \frac{c_1^- - x}{c_1^-} \right)^{1/\alpha}, \quad x \leq c_1^-.$$

Since  $c_1^- \leq 1$  this implies that

$$(9) \quad f_0^{-1}(c) \leq c - c \cdot e^{-\delta/\alpha} (c_1^- - c)^{1/\alpha},$$

and if  $x < c$  then we can use that  $c < c_1^-$  to get

$$(10) \quad f_0^{-1}(x) \leq c - c \cdot e^{-\delta/\alpha} \left(1 - \frac{x}{c}\right)^{1/\alpha}.$$

Using (9) and (10) we get (note that  $f_0^{-1}(c) < c$ ):

$$f_0^{-2}(c) \leq c - c \cdot e^{-\delta/\alpha} \left(1 - \frac{f_0^{-1}(c)}{c}\right)^{1/\alpha} \leq c - c \cdot e^{-\delta(1+\alpha^{-1})/\alpha} (c_1^- - c)^{\alpha^{-2}}.$$

By repeatedly applying (10) to the above inequality we arrive at

$$f_0^{-n}(c) \leq c - c \cdot \exp \left\{ -\frac{\delta}{\alpha} \left(1 + \dots + \alpha^{-(n-1)}\right) \right\} \cdot (c_1^- - c)^{\alpha^{-n}},$$

which together with the fact that  $1 + \dots + \alpha^{-n} < \alpha/(\alpha-1)$  proves (8).

Having proved (7) and (8) we now continue the proof of the lemma. Note that the left-hand side of (7) appears in the right-hand side of (8) and vice versa. Thus we can iterate these inequalities once we have *some* bound for either of them. To this end, suppose  $f_1^{-1}(c) - c \geq t\varepsilon$ , for some  $t > 0$ . If we plug this into (8) and then plug the resulting bound into (7), we get that

$$(11) \quad f_1^{-1}(c) - c \geq \varepsilon \cdot \left(1 - \frac{e^{\delta\alpha/(\alpha-1)} \varepsilon^{1-\alpha^{-b}}}{c t^{\alpha^{-b}}}\right)^{1/\alpha} = \varepsilon h(t).$$

We claim that the map  $h$  has two fixed points: a repeller  $t_0$  close to 0 and an attractor  $t_1$  close to 1. To see this, solve the fixed point equation  $t = h(t)$  to get

$$(12) \quad t^{\alpha^{-b}} (1 - t^\alpha) = \varepsilon^{1-\alpha^{-b}} e^{\delta\alpha/(\alpha-1)} / c.$$

Let  $g(t) = t^{\alpha^{-b}} (1 - t^\alpha)$  and let  $\rho = \varepsilon^{1-\alpha^{-b}} e^{\delta\alpha/(\alpha-1)} / c$ . Note that  $g(0) = 0$ ,  $g(1) = 1$ , and  $g$  has exactly one turning point  $\tau$  at which  $g(\tau) > \rho$  for  $b$  large enough. This

shows that  $g(t) = \rho$  has two solutions  $t_0 < t_1$ . That  $t_0$  is repelling and  $t_1$  attracting (for  $h$ ) follows from the fact that  $h(t)^\alpha \rightarrow -\infty$  as  $t \downarrow 0$  and  $h(t) \rightarrow 1$  as  $t \uparrow \infty$ .

We now find bounds on the fixed points of  $h$ . Solving  $Dg(\tau) = 0$  gives

$$(13) \quad \tau = (\alpha^{b+1} + 1)^{-1/\alpha}.$$

Hence (12) shows that

$$(14) \quad \rho = t_0^{\alpha^{-b}}(1 - t_0^\alpha) > t_0^{\alpha^{-b}}(1 - \tau^\alpha) \implies t_0 < \left( \frac{\rho}{1 - \tau^\alpha} \right)^{\alpha^b}$$

and

$$(15) \quad \rho = t_1^{\alpha^{-b}}(1 - t_1^\alpha) > \tau^{\alpha^{-b}}(1 - t_0^\alpha) \implies t_1 > \left( 1 - (\tau\varepsilon)^{-\alpha^{-b}} \tilde{\rho}\varepsilon \right)^{1/\alpha},$$

where  $\rho = \tilde{\rho}\varepsilon^{1-\alpha^{-b}}$  so that  $\tilde{\rho}$  is a constant not depending on  $b_0$ . By assumption

$$\varepsilon^{-\alpha^{-b}} = (1/\varepsilon)^{\alpha^{-b}} \leq (1/\gamma^{\alpha^{b_0}})^{\alpha^{-b}} \leq 1/\gamma$$

and  $\tau^{-\alpha^{-b}} \rightarrow 1$  as  $b \rightarrow \infty$  by (13), so (15) shows that there exists a constant  $\mu$  such that

$$(16) \quad t_1 > 1 - \mu\varepsilon.$$

All that is need to complete the proof is *some* initial bound  $f_1^{-1}(c) - c \geq t'\varepsilon$  such that  $t' > t_0$ , because then  $h^i(t') \rightarrow t_1$  as  $i \rightarrow \infty$ , which together with (11) and (16) shows that

$$f_1^{-1}(c) - c \geq \varepsilon h(t_1) = \varepsilon t_1 = \varepsilon(1 - \mu\varepsilon).$$

Plugging this into (8) also shows that

$$c - f_0^{-n}(c) \geq c e^{\delta/(\alpha-1)} ((1 - \mu\varepsilon)\varepsilon)^{\alpha^{-n}} > c e^{\delta/(\alpha-1)} (1 - \mu\varepsilon) \varepsilon^{\alpha^{-n}} = c \nu \varepsilon^{\alpha^{-n}}.$$

To get an initial bound  $t'$  we use the fact that  $f_1^{-1}(c) - c > |R|$  and look for a bound on  $|R|$ . Since  $\mathcal{R}f$  is nontrivial we have  $f^{b+1}(R) \supset R$ , which implies

$$|R| \leq |f^b(f(R))| \leq \max_{x < c} f'(x)^b \cdot e^\delta |Q(R)| \leq (e^\delta u \alpha / c)^b e^\delta v (|R|/\varepsilon)^\alpha$$

and thus

$$(17) \quad f_1^{-1}(c) - c > |R| \geq \varepsilon \cdot \left( \frac{c \varepsilon^{1/b}}{\alpha e^{\delta(b+1)/b}} \right)^{b/(\alpha-1)} = \varepsilon t'.$$

Here  $t'$  is of the order  $\varepsilon^{1/(\alpha-1)} \alpha^{-b}$  whereas  $t_0$  is of the order  $\varepsilon^{\alpha^b}$ , so  $t' > t_0$  for  $b_0$  large enough. To see this, solve  $t' > (\rho/(1 - \tau^\alpha))^{\alpha^b}$  for  $\varepsilon$  to get

$$\log \varepsilon < \frac{\alpha - 1}{(\alpha - 1)\alpha^b - \alpha} \cdot \log \left\{ \left( \frac{c}{\alpha e^{\delta(b+1)/b}} \right)^{b/(\alpha-1)} \left( \frac{c(1 - \tau^\alpha)}{e^{\delta\alpha/(\alpha-1)}} \right)^{\alpha^b} \right\}.$$

The right-hand side tends to  $\log\{c e^{-\delta\alpha/(\alpha-1)}\}$  as  $b \rightarrow \infty$ , so it suffices to choose

$$\varepsilon \leq \mathcal{O} \left( \exp \left\{ -\frac{\delta\alpha}{\alpha - 1} \right\} \right)$$

and  $t' > t_0$  will hold (for  $b_0$  large enough).  $\square$

The next lemma is the reason why we chose  $\varepsilon$  to be of the order  $\alpha^{-b_0 \dots}$ . The previous lemma is first used to show that the region where the backward orbit of  $c$  under  $f_0$  is governed by a root behavior is escaped after  $\log b$  steps, and that the remaining  $b - \log b$  steps are then governed by the fixed point at 0. The choice of  $\varepsilon$  will make sure that the linear behavior dominates the root behavior and hence  $c_1^+$  will approach the fixed point at 0 as  $b$  is increased.

**Lemma 4.7.** *There exists  $K$  such that if  $f \in \mathcal{L}_\Omega^1 \cap \mathcal{K}$ , then  $1 - c_1^- < K\varepsilon^2$ . Also,  $c_1^+ \rightarrow 0$  exponentially in  $b_0$  as  $b_0 \rightarrow \infty$ .*

*Remark 4.8.* This lemma also implies that the parameters  $u$  and  $v$  are close to one for  $f \in \mathcal{L}_\Omega^1 \cap \mathcal{K}$  since  $\phi(u) = c_1^-$  and  $\psi(1 - v) = c_1^+$ . Hence, for example

$$1 - u = \phi^{-1}(1) - \phi^{-1}(c_1^-) = |\phi^{-1}([c_1^-, 1])| \leq e^\delta |[c_1^-, 1]| < Ke^\delta \varepsilon^2,$$

and

$$1 - v = \psi^{-1}(c_1^+) - \psi^{-1}(0) = |\psi^{-1}([0, c_1^+])| \leq e^\delta |[0, c_1^+]| \leq K'e^{-b_0},$$

for some  $K'$ .

*Proof.* The proof is based on the fact that  $c_1^+ < f_0^{-b}(c)$  and  $c_1^- > f_1^{-a}(c)$  for monotone combinatorics, so we can use Lemma 4.6 to bound the position of the critical values.

Lemma 4.6 shows that

$$1 - c_1^- < 1 - f_1^{-1}(c) \leq 1 - c - (1 - \mu\varepsilon)\varepsilon = \mu\varepsilon^2,$$

which proves the statement about  $c_1^-$ .

Next, let  $n = \lceil \log_\alpha b_0 \rceil$ . Then  $\alpha^{-n} \leq 1/b_0$  and  $\varepsilon^{\alpha^{-n}} \geq \varepsilon^{1/b_0}$ , so applying Lemma 4.6 again we get

$$\frac{f_0^{-n}(c)}{c} \leq 1 - \nu(\varepsilon^-)^{1/b_0} = 1 - \nu\alpha^{-\sigma}.$$

Thus  $f_0^{-n}(c)$  is a uniform distance away from  $c$ . Since  $b_0 - \lceil \log_\alpha b_0 \rceil \rightarrow \infty$ , and since 0 is an attracting fixed point for  $f_0^{-1}$  with uniform bound on the multiplier, it follows that  $f_0^{-b}(c)$  approaches 0 exponentially as  $b_0 \rightarrow \infty$ . This proves the statement about  $c_1^+$ .  $\square$

Now that we have control over the critical values we can estimate the derivative of the return map. The derivative of  $f_1^a$  is easy to control since  $f_1^a$  is basically a linear map on a neighborhood of  $f(L)$ . However, the derivative of  $f_0^b$  is a bit more delicate and we are only able to estimate it on a subset of  $f(R)$ . The idea is to split the derivative calculation into two regions; one expanding region governed by the fixed point at 0 and one contracting region in the vicinity of the critical point. The choice of  $\varepsilon$  will ensure that the expanding region dominates the contracting region if  $b$  is sufficiently large.

We will need the following expressions for the derivatives of the inverse branches of  $f$ :

$$(18) \quad Df_0^{-1}(x) = \frac{c}{\alpha} \cdot \frac{D\phi^{-1}(x)}{u} \left( \frac{|\phi^{-1}([0, c_1^-])|}{|\phi^{-1}([x, c_1^-])|} \right)^{1-1/\alpha},$$

$$(19) \quad Df_1^{-1}(x) = \frac{\varepsilon}{\alpha} \cdot \frac{D\psi^{-1}(x)}{v} \left( \frac{|\psi^{-1}([c_1^+, 1])|}{|\psi^{-1}([c_1^+, x])|} \right)^{1-1/\alpha}.$$

The above equations can be derived from (5) and (6).

**Lemma 4.9.** *There exists  $K$  such that if  $f \in \mathcal{L}_\Omega^1 \cap \mathcal{K}$ , then*

$$\begin{aligned} K^{-1}(\varepsilon/\alpha)^a &\leq Df_1^{-a}(x) \leq K(\varepsilon/\alpha)^a, & \forall x > f_0^{-1}(c), \\ K^{-1}\alpha^{-b}\varepsilon^{-1+\alpha^{-b}} &\leq Df_0^{-b}(c) \leq K\alpha^{-b}\varepsilon^{-1+\alpha^{-b}}. \end{aligned}$$

*Proof.* We start by proving the lower bound on  $Df_1^{-a}$ . From (19) we get  $Df_1^{-1}(x) \geq e^{-\delta}\varepsilon/\alpha$  and hence

$$Df_1^{-a}(x) \geq e^{-a\delta}(\varepsilon/\alpha)^a, \quad \forall x \in [c_1^+, 1].$$

Note that  $e^{-a\delta}$  has a lower bound that does not depend on  $b_0$ , so the above equation shows that  $Df_1^{-a}(x) \geq K^{-1}(\varepsilon/\alpha)^a$  for some  $K$  (not depending on  $b_0$ ).

Next consider the upper bound on  $Df_1^{-a}$ . Use  $c_1^- \leq 1$  and (5) to see that

$$(20) \quad f_0^{-1}(c) \geq c \left(1 - (e^\delta \varepsilon)^{1/\alpha}\right).$$

Equation (19), the fact that  $1 - c_1^+ \leq 1$ , and the assumption that  $x > f_0^{-1}(c)$  together imply that

$$(21) \quad Df_1^{-1}(x) \leq \frac{\varepsilon e^\delta}{\alpha v} \left( e^\delta \frac{1 - c_1^+}{x - c_1^+} \right)^{1-1/\alpha} \leq \frac{e^\delta}{v} \left( \frac{e^\delta}{f_0^{-1}(c) - c_1^+} \right)^{1-1/\alpha} \cdot \frac{\varepsilon}{\alpha}$$

Equation (20) and Lemma 4.7 show that  $f_0^{-1}(c) - c_1^+$  has a lower bound that is independent of  $b_0$  and Remark 4.8 can be used to bound  $v$ . Hence the expression in front of  $\varepsilon/\alpha$  in (21) has an upper bound that does not depend on  $b_0$ . Since  $x > f_1^{-1}(c)$  implies that  $f_1^{-i}(x) > f_0^{-1}(c)$  for all  $i = 1, \dots, a$ , the previous argument and (21) shows that

$$Df_1^{-a}(x) \leq K(\varepsilon/\alpha)^a,$$

for some  $K$  (not depending on  $b_0$ ).

We now turn to proving the bounds on  $Df_0^{-b}(c)$ . Equation (18) shows that

$$(22) \quad \frac{ce^{-\delta}}{\alpha} \left( e^{-\delta} \frac{c_1^-}{c_1^- - x} \right)^{1-1/\alpha} \leq Df_0^{-1}(x) \leq \frac{e^\delta}{\alpha u} \left( e^\delta \frac{c_1^-}{c_1^- - x} \right)^{1-1/\alpha}$$

The upper bound in (22) gives

$$(23) \quad Df_0^{-b}(x) = \prod_{i=0}^{b-1} Df_0^{-1}(f_0^{-i}(x)) \leq \left( \frac{e^{2\delta}}{\alpha u} \right)^b \cdot \prod_{i=0}^{b-1} \left( \frac{c_1^-}{c_1^- - f_0^{-i}(x)} \right)^{1-1/\alpha}$$

The expression before the last product is bounded by  $K\alpha^{-b}$  for some constant  $K$  since: (i)  $b\delta \leq (1 + (\sigma/\alpha)^2 - \beta)b_0/(b_0)^2 \rightarrow 0$  by (4), and (ii)  $u^b \geq (1 - \mathcal{O}(\varepsilon^2))^b$  by Remark 4.8 and  $b\varepsilon \leq b\theta\alpha^{-b_0\sigma/\alpha^2} \rightarrow 0$ , so  $u^b$  has a lower bound which does not depend on  $b$ .

The lower bound in (22) similarly shows that

$$(24) \quad Df_0^{-b}(x) \geq \left( \frac{c}{\alpha e^{2\delta}} \right)^b \cdot \prod_{i=0}^{b-1} \left( \frac{c_1^-}{c_1^- - f_0^{-i}(x)} \right)^{1-1/\alpha}$$

The expression before the product is bounded by  $K^{-1}\alpha^{-b}$  for some constant  $K$  since  $b\delta \rightarrow 0$  as in (i) above, and  $c^b \geq (1 - \theta\alpha^{-b_0\sigma/\alpha^2})^b$  so  $c^b$  has a lower bound independent of  $b$ .



The product in (23) and (24) is the same, so we will look for bounds on this product next. We claim that there exists constants  $\gamma, \rho > 0$  such that

$$(25) \quad e^{-\delta/(\alpha-1)} \leq \frac{c_1^- - f_0^{-n}(x)}{c_1^-} \cdot \left( \frac{c_1^- - x}{c_1^-} \right)^{-\alpha^{-n}} \leq e^{\delta/(\alpha-1)} (1 + \gamma \varepsilon^\rho)^{\alpha/(\alpha-1)},$$

for  $x \leq c$ . Assume that this holds for the moment (we will prove it shortly).

Equations (23) and (25) show that

$$Df_0^{-b}(c) \leq \frac{K}{\alpha^b} \left\{ \prod_{i=0}^{b-1} e^{\delta/(\alpha-1)} \left( \frac{c_1^- - c}{c_1^-} \right)^{-\alpha^{-i}} \right\}^{1-1/\alpha} = \frac{K e^{b\delta/\alpha}}{\alpha^b} \left( \frac{c_1^- - c}{c_1^-} \right)^{-1+\alpha^{-b}}.$$

(The equality follows from a computation using the fact that the logarithm of the above product is a geometric sum.) From Lemma 4.7 we get

$$\left( \frac{c_1^- - c}{c_1^-} \right)^{-1+\alpha^{-b}} \leq \left( \varepsilon \frac{1 - K_1 \varepsilon}{1 - K_1 \varepsilon^2} \right)^{-1+\alpha^{-b}} \leq K_2 \varepsilon^{-1+\alpha^{-b}},$$

and hence

$$Df_0^{-b}(c) \leq \frac{K_3 e^{b\delta/\alpha}}{\alpha^b} \varepsilon^{-1+\alpha^{-b}}.$$

Since  $b\delta \rightarrow 0$  this finishes the proof of the upper bound on  $Df_0^{-b}(c)$ .

Similarly, (24) and (25) show that

$$Df_0^{-b}(c) \geq \left( K \alpha^b e^{b\delta/\alpha} (1 + \gamma \varepsilon^\rho)^b \right)^{-1} \left( \frac{c_1^- - c}{c_1^-} \right)^{-1+\alpha^{-b}}.$$

Use  $c_1^- < 1$  to get  $(c_1^- - c)/c_1^- = 1 - c/c_1^- < 1 - c = \varepsilon$ . This finishes the proof of the lower bound of  $Df_0^{-b}(c)$ , since: (i)  $b\delta \rightarrow 0$ , and (ii)  $(1 + \gamma \varepsilon^\rho)^b \rightarrow 1$  since  $b\varepsilon^\rho \leq b\theta \alpha^{-\rho b_0 \sigma / \alpha^2} \rightarrow 0$  for any  $\rho > 0$ .

It only remains to prove the claim (25). We start with the lower bound. From (5) and  $c < c_1^-$  (the latter follows from Lemma 4.7) we get that

$$\frac{c_1^- - f_0^{-1}(x)}{c_1^-} > \frac{c - f_0^{-1}(x)}{c} \geq e^{-\delta/\alpha} \left( \frac{c_1^- - x}{c_1^-} \right)^{1/\alpha}.$$

Hence

$$\frac{c_1^- - f_0^{-2}(x)}{c_1^-} \geq e^{-\delta/\alpha} \left( \frac{c_1^- - f_0^{-1}(x)}{c_1^-} \right)^{1/\alpha} \geq e^{-\delta(1+1/\alpha)/\alpha} \left( \frac{c_1^- - x}{c_1^-} \right)^{\alpha^{-2}}.$$

so an induction argument and  $1 + \dots + 1/\alpha^{n-1} < \alpha/(\alpha-1)$  finishes the proof of the lower bound of (25).

We finally prove the upper bound of (25). From  $c < c_1^- < 1$  and (5) we get

$$(26) \quad \frac{c}{c_1^-} \frac{c_1^- - f_0^{-1}(x)}{c - f_0^{-1}(x)} \leq \frac{(1-c) + (c - f_0^{-1}(x))}{c - f_0^{-1}(x)} \leq 1 + \frac{\varepsilon e^{\delta/\alpha}}{c(c_1^- - x)^{1/\alpha}}.$$

Assuming that  $x \leq c$  we get  $c_1^- - x \geq c_1^- - c$ .<sup>5</sup> By Lemma 4.7,  $c_1^- - c \geq k\varepsilon$  which together with (5) and (26) shows that

$$\frac{c_1^- - f_0^{-1}(x)}{c_1^-} \leq \left(1 + \frac{e^{\delta/\alpha}}{ck^{1/\alpha}}\varepsilon^{1-1/\alpha}\right) \frac{c - f_0^{-1}(x)}{c} \leq (1 + \gamma\varepsilon^\rho)e^{\delta/\alpha} \left(\frac{c_1^- - x}{c_1^-}\right)^{1/\alpha},$$

for some constant  $\gamma > 0$  and  $\rho = 1 - 1/\alpha > 0$ . This shows that

$$\begin{aligned} \frac{c_1^- - f_0^{-2}(x)}{c_1^-} &\leq (1 + \gamma\varepsilon^\rho)e^{\delta/\alpha} \left(\frac{c_1^- - f_0^{-1}(x)}{c_1^-}\right)^{1/\alpha} \\ &\leq \left((1 + \gamma\varepsilon^\rho)e^{\delta/\alpha}\right)^{1+1/\alpha} \left(\frac{c_1^- - x}{c_1^-}\right)^{\alpha^{-2}}, \end{aligned}$$

so an induction argument and  $1 + \dots + 1/\alpha^{n-1} < \alpha/(\alpha - 1)$  finishes the proof of the upper bound of (25).  $\square$

Armed with the above lemmas we can start proving invariance. The first step is to show that  $\varepsilon(\mathcal{R}f)$  is small. The proof is complicated by the fact that we do not know anything about  $\varepsilon(\mathcal{R}f)$ . Once we find some bound on  $\varepsilon(\mathcal{R}f)$  we can show that it in fact is very small.

**Proposition 4.10.** *There exists  $\theta > 0$  (not depending on  $b_0$ ) such that*

$$f \in \mathcal{L}_\Omega^S \cap \mathcal{K} \implies \varepsilon(\mathcal{R}f) \leq \theta\alpha^{-b_0\sigma/\alpha^2},$$

for  $b_0$  large enough.

*Proof.* First we find an upper bound on  $|R|$ . Since  $f$  is renormalizable  $f^b(f(R)) \subset C$ . By the mean value theorem there exists  $\xi \in f(R)$  such that  $Df^b(\xi)|f(R)| = |f^b(f(R))|$ . We estimate  $|f(R)| \geq e^{-\delta}|Q_1(R)| \geq e^{-\delta}v(|R|/\varepsilon)^\alpha$ . Taken all together we get

$$(27) \quad |R|^\alpha \leq \frac{e^\delta \varepsilon^\alpha |f(R)|}{v} = \frac{e^\delta \varepsilon^\alpha |f^b(f(R))|}{v Df^b(\xi)} \leq \frac{e^\delta \varepsilon^\alpha |C|}{v Df^b(\xi)}, \quad \xi \in f(R).$$

Next, we find a lower bound on  $|L|$ . Since  $f$  is renormalizable and  $\mathcal{R}f$  is nontrivial we get that  $f^a(f(L)) \supset L$ . By the mean value theorem there exists  $\eta \in f(L)$  such that  $|f^a(f(L))| = Df^a(\eta)|f(L)|$ . Use these two facts to estimate

$$|L| \leq |f^a(f(L))| = Df^a(\eta)|f(L)| \leq Df^a(\eta)e^\delta u(|L|/c)^\alpha,$$

and hence

$$(28) \quad |L|^{\alpha-1} \geq \frac{c^\alpha}{e^\delta u Df^a(\eta)}, \quad \eta \in f(L).$$

There are now two cases to consider: either  $|L| < |R|$  or  $|L| \geq |R|$ . The former case will turn out not to hold, but we do not know that yet.

*Case 1:* In order to reach a contradiction, we assume that  $|L| < |R|$ . This implies that  $|C| < 2|R|$  so equations (27) and (28) show that

$$(29) \quad \frac{|R|}{|L|} \leq \mathcal{O} \left( \varepsilon^\alpha \frac{Df^a(\eta)}{Df^b(\xi)} \right)^{1/(\alpha-1)}.$$

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<sup>5</sup>The condition  $x \leq c$  is unnecessarily strong here; we could get away with  $c_1^- - x \geq k\varepsilon^t$  for some constant  $t$  close to (but smaller than)  $\alpha$ .

We would like to apply Lemma 4.9, but we do not know the position of  $f^b(\xi)$  in relation to  $c$ . However, we claim that the distortion of  $f^b$  on  $f(R)$  is very small which will allow us to use Lemma 4.9 anyway, since

$$Df^b(\xi) = \frac{Df^b(\xi)}{Df^b(f_0^{-b}(c))} \frac{1}{Df_0^{-b}(c)} \geq \left( Df_0^{-b}(c) \cdot \exp \{ \text{Dist } f^b|_{f(R)} \} \right)^{-1}$$

Note that  $f^a(\eta) > f_0^{-1}(c)$  since  $f$  is renormalizable, so we can directly apply Lemma 4.9 to estimate  $Df^a(\eta)$ .

We now prove that the distortion of  $f^b$  on  $f(R)$  is small. For monotone combinatorics we have

$$f(R) \subset (f_0^{-b-1}(c), f_0^{-b+1}(c)),$$

thus

$$|f_0^{-b+1}(c) - f_0^{-b-1}(c)| \geq |f(R)| \geq \varepsilon^{-\delta} v(|R|/\varepsilon)^\alpha,$$

and consequently

$$\frac{|R|}{\varepsilon} \leq \left( \frac{e^\delta}{v} \cdot |f_0^{-b+1}(c) - f_0^{-b-1}(c)| \right)^{1/\alpha} \rightarrow 0, \quad \text{as } b_0 \rightarrow \infty.$$

This and Lemma 4.7 shows that the length of the right component of  $[0, c_1^-] \setminus C$  is much larger than  $C$  (since  $|R| > |C|/2$  by assumption). Since  $f^{-b}|_C$  extends monotonously to  $[0, c_1^-]$  the Koebe lemma implies that the distortion of  $f^b|_{f(R)}$  tends to zero as  $b_0 \rightarrow \infty$ . (Note that the left component of  $[0, c_1^-] \setminus C$  is of order 1 so it is automatically large compared to  $C$ .)

Now that we have control over the distortion, apply Lemma 4.9 to get

$$\varepsilon^\alpha \frac{Df^a(\eta)}{Df^b(\xi)} = \mathcal{O} \left( \varepsilon^{-(a+1-\alpha-\alpha^{-b})} \alpha^{-(b-a)} \right).$$

By (3)  $a+1-\alpha-\alpha^{-b} \geq \sigma - \alpha^{-b}$  and we may assume that  $\sigma > \alpha^{-b}$  (by choosing  $b_0$  sufficiently large) so that the exponent of  $\varepsilon$  is negative. Inserting  $\varepsilon \geq \alpha^{-b_0/\alpha}$  we get that the right-hand side is at most of the order  $\alpha^{-t}$ , where

$$t = -\frac{b_0}{\alpha}(\alpha - \sigma - \alpha^{-b}) + b_0 = \left( \frac{\sigma + \alpha^{-b}}{\alpha} \right) b_0.$$

The expression in front of  $b_0$  is positive so  $t \rightarrow \infty$  as  $b_0 \rightarrow \infty$ . Hence (29) shows that  $|R|/|L| \rightarrow 0$  as  $b_0 \rightarrow \infty$ . This contradicts the assumption that  $|R| > |L|$ , so we conclude that  $|R| \leq |L|$ .

*Case 2:* From the argument above we know that  $|L| \geq |R|$ . In particular,  $|C| \leq 2|L|$ , so equations (27) and (28) show that

$$(30) \quad \frac{|R|}{|L|} \leq \frac{\varepsilon}{c} \left( \frac{2e^{2\delta} u}{v} \frac{Df^a(\eta)}{Df^b(\xi)} \right)^{1/\alpha}.$$

As in Case 1 we would like to apply Lemma 4.9 but first we need to show that the distortion of  $f^b$  on  $f(R)$  is small. In order to do so we need an upper bound on  $|L|$ .

Since  $f$  is renormalizable  $f(L) \subset C$ , so another mean value theorem estimate gives

$$2|L| \geq |C| \geq Df^a(\zeta)|f(L)| \geq Df^a(\xi)e^{-\delta}u(|L|/c)^\alpha,$$

for some  $\zeta \in f(L)$ . Now apply Lemma 4.9 to get that

$$(31) \quad |L| \leq \mathcal{O} \left( \varepsilon^{a/(\alpha-1)} \right).$$

By (3)  $a \geq \alpha - 1 + \sigma$  so once again we get that the length of the right component of  $[0, c_1^-] \setminus C$  is large compared to  $C$  (use Lemma 4.7 to bound  $c_1^-$ ). The Koebe lemma shows that the distortion of  $f^b|_{f(R)}$  tends to zero as  $b_0 \rightarrow \infty$ .

We can now apply Lemma 4.9 to (30) to get that

$$(32) \quad \varepsilon(\mathcal{R}f) = \frac{|R|}{|L| + |R|} < \frac{|R|}{|L|} \leq K \left( \varepsilon^{-(a+1-\alpha-\alpha^{-b})} \alpha^{-(b-a)} \right)^{1/\alpha}.$$

As in the above we may assume that the exponent of  $\varepsilon$  is negative, so inserting  $\varepsilon \geq \alpha^{-b_0/\alpha}$  we get

$$\varepsilon(\mathcal{R}f) \leq K_1 \left( \alpha^{b_0(a+1-\alpha-\alpha^{-b})/\alpha-b_0} \right)^{1/\alpha} \leq K_2 \left( \alpha^{b_0(\alpha-\sigma)/\alpha-b_0} \right)^{1/\alpha} = K_2 \alpha^{-b_0\sigma/\alpha^2}.$$

Let  $\theta = K_2$  to finish the proof.  $\square$

Knowing that  $\varepsilon(\mathcal{R}f)$  is small it is relatively straightforward to use the Koebe lemma to prove that the distortion of the diffeomorphic parts of  $\mathcal{R}f$  is small. Here we really need the condition that the return time of the left branch satisfies  $a > \alpha - 1$  in order to find some Koebe space. Also note that we assume negative Schwarzian derivative so that we can apply the strong version of the Koebe lemma (see Lemma C.4) which gives explicit bounds on the distortion.

**Proposition 4.11.** *If  $f \in \mathcal{L}_\Omega^S \cap \mathcal{K}$ , then  $\text{Dist } \phi(\mathcal{R}f) \leq \delta$  and  $\text{Dist } \psi(\mathcal{R}f) \leq \delta$ , for  $b_0$  large enough.*

*Proof.* From Proposition 4.10 we know that  $|L| > |R|$  and thus (31) applies, which shows that  $|C|$  is at most of the order  $\varepsilon^{a/(\alpha-1)}$ . Hence Lemma 4.7 shows that the right component of  $(c_1^+, c_1^-) \setminus C$  has length of order  $\varepsilon$  and the left component has length of order 1.

Let  $\hat{U} = f_1^{-a}(C)$  and  $\hat{V} = f_0^{-b}(C)$ . The inverses of  $f^a|_{\hat{U}}$  and  $f^b|_{\hat{V}}$  extend monotonously (at least) to  $(c_1^+, c_1^-)$  so the Koebe lemma (see Corollary C.5) implies that the distortion of these maps is of the order  $\varepsilon^t$ , where

$$t = -1 + a/(\alpha - 1) > \sigma/(\alpha - 1) > 0$$

by (3).

Since  $\phi(\mathcal{R}f)$  equals  $f^a|_{\hat{U}} \circ \phi$  and  $\psi(\mathcal{R}f)$  equals  $f^b|_{\hat{V}} \circ \psi$  (up to rescaling) this shows that

$$(33) \quad \text{Dist } \phi(\mathcal{R}f) \leq K\varepsilon^t \quad \text{and} \quad \text{Dist } \psi(\mathcal{R}f) \leq K\varepsilon^t.$$

Note that  $\varepsilon^t \leq \theta \alpha^{-tb_0\sigma/\alpha^2} \ll \delta$  for  $b_0$  large enough, since  $\delta = (1/b_0)^2$  by (4).  $\square$

The final step in the invariance proof is showing that  $\varepsilon(\mathcal{R}f)$  is not too small. This is the only place where we use the condition on  $c_1^+(\mathcal{R}f)$ . This condition excludes maps whose renormalization has a trivial right branch. Such maps are difficult for us to handle because  $\varepsilon(\mathcal{R}f)$  may be smaller than the lower bound on  $\varepsilon$ .

**Proposition 4.12.** *If  $f \in \mathcal{L}_\Omega^S \cap \mathcal{K}$  and if  $1 - c_1^+(\mathcal{R}f) \geq \lambda$  for some  $\lambda > 0$  not depending on  $b_0$ , then  $\varepsilon(\mathcal{R}f) \geq \alpha^{-b_0/\alpha}$ , for  $b_0$  large enough.*

*Proof.* First we look for a lower bound on  $|R|$ . The condition on  $c_1^+(\mathcal{R}f)$  gives

$$1 - \lambda \geq c_1^+(\mathcal{R}f) = 1 - \frac{|f^b(f(R))|}{|C|}$$

and hence  $|f^b(f(R))| \geq \lambda|C| \geq \lambda|L|$ .<sup>6</sup> On the other hand, the mean value theorem shows that there exists  $\xi \in f(R)$  such that

$$|f^b(f(R))| = Df^b(\xi)|f(R)| \leq Df^b(\xi)e^\delta(|R|/\varepsilon)^\alpha.$$

Thus

$$(34) \quad |R|^\alpha \geq \frac{\lambda|L|\varepsilon^\alpha}{e^\delta Df^b(\xi)}, \quad \xi \in f(R).$$

Next, we look for an upper bound on  $|L|$ . The mean value theorem in conjunction with  $C \supset f^a(f(L))$  and  $2|L| > |C|$ , shows that  $2|L| > |C| \geq Df^a(\eta)e^{-\delta}u(|L|/c)^\alpha$ , for some  $\eta \in f(L)$ . Hence

$$(35) \quad |L|^{\alpha-1} \leq \frac{2e^\delta c^\alpha}{uDf^a(\eta)}, \quad \eta \in f(L).$$

Equations (34) and (35) show that

$$(36) \quad \frac{|R|}{|L|} \geq \frac{\varepsilon}{c} \left( \frac{\lambda u}{2e^{2\delta}} \frac{Df^a(\eta)}{Df^b(\xi)} \right)^{1/\alpha}.$$

Now apply Lemma 4.9 (Proposition 4.11 can be used to bound  $Df^b(\xi)$  in case  $f^b(\xi) > c$ ) to get that

$$\varepsilon(\mathcal{R}f) = \frac{|R|}{|L| + |R|} = \frac{|R|}{|L|} \cdot \left( 1 + \frac{|R|}{|L|} \right)^{-1} \geq k_0 \left( \varepsilon^{-(a+1-\alpha-\alpha^{-b})} \alpha^{-(b-a)} \right)^{1/\alpha}.$$

By (3),  $a + 1 - \alpha - \alpha^{-b} \geq \sigma - \alpha^{-b}$  which we may assume to be positive (by choosing  $b_0$  large), so inserting  $\varepsilon \leq \theta \alpha^{-b_0 \sigma / \alpha^2}$  in the right-hand side we get that  $\varepsilon(\mathcal{R}f) \geq k_1 \alpha^{-t/\alpha}$ , where

$$(37) \quad t = -\frac{b_0 \sigma}{\alpha^2} (\sigma - \alpha^{-b_0}) + (1 + (\sigma/\alpha)^2 - \beta) b_0 = \left( 1 + \frac{\sigma \alpha^{-b_0}}{\alpha^2} - \beta \right) b_0.$$

We may assume that  $\beta > \sigma \alpha^{-b_0} / \alpha^2$  by choosing  $b_0$  large enough. Hence

$$\varepsilon(\mathcal{R}f) \geq k_1 \alpha^{-\rho b_0 / \alpha}, \quad \rho = 1 + \frac{\sigma \alpha^{-b_0}}{\alpha^2} - \beta,$$

which is larger than  $\varepsilon^{-b_0/\alpha}$  for  $b_0$  large enough since  $\rho \downarrow 1 - \beta < 1$  as  $b_0 \rightarrow \infty$ .  $\square$

The above propositions are all we need to prove invariance:

*Proof of Theorem 4.2.* Apply Propositions 4.10, 4.11 and 4.12.  $\square$

To prove Theorem 4.3 we need to show that twice renormalizable maps in  $\mathcal{K}$  automatically satisfy the condition on  $c_1^+(\mathcal{R}f)$ . The only problem is that twice renormalizable maps may have  $\varepsilon(\mathcal{R}f) < \alpha^{-b_0/\alpha}$  in general, but even so we can still apply Lemma 4.6 to  $\mathcal{R}f$  to get some bound on  $c_1^+(\mathcal{R}f)$ .

*Proof of Theorem 4.3.* If we go through the proof of Proposition 4.12 without using the condition on  $c_1^+(\mathcal{R}f)$  and instead use  $f^b(f(R)) \supset R$ , then (34) becomes

$$|R|^{\alpha-1} \geq \frac{\varepsilon^\alpha}{e^\delta Df^b(\xi)}, \quad \xi \in f(R),$$

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<sup>6</sup>This is the *only* place where we use the condition on  $c_1^+(\mathcal{R}f)$ .

and (36) becomes

$$\frac{|R|}{|L|} \geq \frac{\varepsilon}{c} \left( \frac{u}{2e^{2\delta}} \frac{Df^a(\eta)}{Df^b(\xi)} \right)^{1/(\alpha-1)}.$$

This time we get that  $\varepsilon(\mathcal{R}f) \geq k_1 \alpha^{-t/(\alpha-1)}$ , where  $t$  is the same as in (37). However, the important thing to note is that we still get a lower bound of the type  $\tilde{\varepsilon} = \varepsilon(\mathcal{R}f) \geq k_1 \alpha^{-Kb_0}$ . This and Propositions 4.10 and 4.11 show that we can apply Lemma 4.6 to  $\tilde{f} = \mathcal{R}f$ .

By the above argument we can apply Lemma 4.6 to  $\tilde{f}$  and  $\tilde{\varepsilon} = 1 - \tilde{c}$  to get that

$$\frac{\tilde{c} - \tilde{f}_0^{-b}(\tilde{c})}{\tilde{c}} \geq \nu (k_1 \alpha^{-Kb_0})^{\alpha-b} \geq k_2 > 0.$$

Let  $\lambda = k_2$ . Note that for monotone combinatorics  $c_1^+ \leq f_0^{-b}(c)$  and since  $\tilde{f}$  is renormalizable this shows that  $1 - c_1^+(\tilde{f}) > \lambda$ .  $\square$

## 5. A PRIORI BOUNDS

In this section we begin exploiting the existence of the relatively compact ‘invariant’ set of Theorem 4.2. An important consequence of this theorem is the existence of so-called *a priori bounds* (or *real bounds*) for infinitely renormalizable maps. We use the a priori bounds to analyze infinitely renormalizable maps and their attractors.

From now on we will assume that the sets  $\Omega$  and  $\mathcal{K}$  of Definition 4.1 have been fixed; in particular, we assume that  $b_0$  has been chosen large enough for Theorem 4.3 to hold.

**Theorem 5.1** (A priori bounds). *If  $f \in \mathcal{L}_\omega^S \cap \mathcal{K}$  is infinitely renormalizable with  $\bar{\omega} \in \Omega^\mathbb{N}$ , then  $\{\mathcal{R}^n f\}_{n \geq 0}$  is a relatively compact family (in  $\mathcal{L}^0$ ).*

*Proof.* This is a consequence of Theorem 4.3 and Proposition 4.5.  $\square$

**Theorem 5.2.** *If  $f \in \mathcal{L}_\omega^S \cap \mathcal{K}$  is infinitely renormalizable with  $\bar{\omega} \in \Omega^\mathbb{N}$ , then  $f$  satisfies the weak Markov property.*

*Proof.* Since  $f$  is infinitely renormalizable there exists a sequence  $C_0 \supset C_1 \supset \dots$  of nice intervals whose lengths tend to zero (i.e.  $C_n$  is the range of the  $n$ -th first-return map and this interval is nice since the boundary consists of periodic points whose orbits do not enter  $C_n$ ).

Let  $T_n$  denote the transfer map to  $C_n$ . We must show that  $T_n$  is defined almost everywhere and that there exists  $\delta > 0$  (not depending on  $n$ ) such that  $C_n$  contains a  $\delta$ -scaled neighborhood of  $C_{n+1}$ , for every  $n \geq 0$ .

By a theorem of Singer<sup>7</sup>  $f$  cannot have a periodic attractor since it would attract at least one of the critical values. This does not happen for infinitely renormalizable maps since the critical orbits have subsequences which converge on the critical point. Thus Proposition 3.7 shows that  $T_n$  is defined almost everywhere.

Let  $L_n = C_n \cap (0, c)$  and let  $R_n = C_n \cap (c, 1)$ , where  $c$  is the critical point of  $f$ . Since  $f$  is infinitely renormalizable there exists  $l_n$  and  $r_n$  such that  $f^{l_n}(L_n)$  is in

<sup>7</sup> Singer’s theorem is stated for unimodal maps but the statement and proof can easily be adapted to Lorenz maps.

the right component of  $C_{n-1} \setminus C_n$ , and such that  $f^{r_n}(R_n)$  is contained in the left component of  $C_{n-1} \setminus C_n$ . We contend that

$$(38) \quad \inf_n |f^{l_n}(L_n)|/|C_n| > 0 \quad \text{and} \quad \inf_n |f^{r_n}(R_n)|/|C_n| > 0.$$

Suppose not, and consider the  $\mathcal{C}^0$ -closure of  $\{\mathcal{R}^n f\}$ . The a priori bounds show that this set is compact and hence there exists a subsequence  $\{\mathcal{R}^{n_k} f\}$  which converges to some  $f_*$ . But then  $f_*$  is a renormalizable map whose cycles of renormalization contain an interval of zero diameter. This is impossible, hence (38) must hold.

Equation (38) shows that  $C_{n-1}$  contains a  $\delta$ -scaled neighborhood of  $C_n$  and that  $\delta$  does not depend on  $n$ .  $\square$

**Theorem 5.3.** *Assume  $f \in \mathcal{L}_\omega^S \cap \mathcal{K}$  is infinitely renormalizable with  $\bar{\omega} \in \Omega^\mathbb{N}$ . Let  $\Lambda$  be the closure of the orbits of the critical values. Then:*

- $\Lambda$  is a Cantor set,
- $\Lambda$  has Lebesgue measure zero,
- the Hausdorff dimension of  $\Lambda$  is strictly inside  $(0, 1)$ ,
- the complement of the basin of attraction of  $\Lambda$  has zero Lebesgue measure.

*Proof.* Let  $L_n$  and  $R_n$  denote the left and right half of the return interval of the  $n$ -th first-return map, let  $i_n$  and  $j_n$  be the return times for  $L_n$  and  $R_n$ , let  $\Lambda_0 = [0, 1]$ , and let

$$\Lambda_n = \bigcup_{i=0}^{i_n-1} \text{cl } f^i(L_n) \cup \bigcup_{j=0}^{j_n-1} \text{cl } f^j(R_n), \quad n = 1, 2, \dots$$

Components of  $\Lambda_n$  are called intervals of generation  $n$  and components of  $\Lambda_{n-1} \setminus \Lambda_n$  are called gaps of generation  $n$  (see Figure 4).

Let  $I$  be an interval of generation  $n$ , let  $J \subset I$  be an interval of generation  $n+1$ , and let  $G \subset I$  be a gap of generation  $n+1$ . We claim that there exists constants  $0 < \mu < \lambda < 1$  such that

$$\mu < |J|/|I| < \lambda \quad \text{and} \quad \mu < |G|/|I| < \lambda,$$

where  $\mu$  and  $\lambda$  do not depend on  $I$ ,  $J$  and  $G$ . To see this, take the  $\mathcal{L}^0$ -closure of  $\{\mathcal{R}^n f\}$ . This set is compact in  $\mathcal{L}^0$ , so the infimum and supremum of  $|J|/|I|$  over all  $I$  and  $J$  as above are bounded away from 0 and 1 (otherwise there would exist an infinitely renormalizable map in  $\mathcal{L}^0$  with  $I$  and  $J$  as above such that  $|J| = 0$  or  $|I| = |J|$ ). The same argument holds for  $I$  and  $G$ . Since  $\{\mathcal{R}^n f\}$  is a subset of the closure the claim follows.

Next we claim that  $\Lambda = \bigcap \Lambda_n$ . Clearly  $\Lambda \subset \bigcap \Lambda_n$  (since the critical values are contained in the closure of  $f(L_n) \cup f(R_n)$  for each  $n$ ). From the previous claim  $|\Lambda_n| < \lambda |\Lambda_{n-1}|$  so the lengths of the intervals of generation  $n$  tend to 0 as  $n \rightarrow \infty$ . Hence  $\Lambda = \bigcap \Lambda_n$ .

It now follows from standard arguments that  $\Lambda$  is a Cantor set of zero measure with Hausdorff dimension in  $(0, 1)$ .

It only remains to prove that almost all points are attracted to  $\Lambda$ . Let  $T_n$  denote the transfer map to the  $n$ -th return interval  $C_n$ . By Proposition 3.7 the domain of  $T_n$  has full measure for every  $n$  and hence almost every point visits every  $C_n$ . This finishes the proof.  $\square$

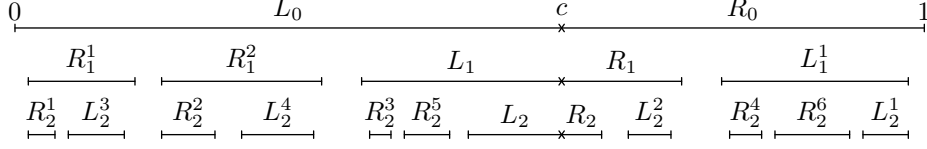


FIGURE 4. Illustration of the intervals of generations 0, 1 and 2 for a (01, 100)-renormalizable map. Here  $L_n^i = f^i(L_n)$  and  $R_n^i = f^i(R_n)$ . The intersection of all levels  $n = 0, 1, 2, \dots$  is a Cantor set, see Theorem 5.3.

## 6. PERIODIC POINTS OF THE RENORMALIZATION OPERATOR

In this section we prove the existence of periodic points of the renormalization operator. The argument is topological and does not imply uniqueness even though we believe the periodic points to be unique within each combinatorial class.<sup>8</sup>

The notation used here is the same as in Section 4, in particular the sets  $\Omega$  and  $\mathcal{K}$  are defined in Definition 4.1. We will implicitly assume that  $b_0$  has been chosen large enough for Theorem 4.2 to hold.

**Theorem 6.1.** *For every periodic combinatorial type  $\bar{\omega} \in \Omega^{\mathbb{N}}$  there exists a periodic point of  $\mathcal{R}$  in  $\mathcal{L}_{\bar{\omega}}$ .*

*Remark 6.2.* We are not saying anything about the periods of the periodic points. For example, we are *not* asserting that there exists a period-two point of type  $(\omega, \omega)^{\infty}$  for some  $\omega \in \Omega$  — all we say is that there is a fixed point of type  $(\omega)^{\infty}$ . The point here is that  $(\omega, \omega)^{\infty}$  is just another way to write  $(\omega)^{\infty}$  so these two types are the same.

To begin with we will consider the restriction  $\mathcal{R}_{\omega}$  of  $\mathcal{R}$  to some  $\omega \in \Omega$  and show that  $\mathcal{R}_{\omega}$  has a fixed point. Fix  $\lambda \in (0, 1)$  and let

$$\mathcal{Y} = \mathcal{L}_{\omega}^S \cap \mathcal{K}, \quad \text{and} \quad \mathcal{Y}_{\lambda} = \{f \in \mathcal{Y} \mid 1 - c_1^+(\mathcal{R}f) \geq \lambda\}.$$

Note that if  $\lambda$  is made smaller then we may have to compensate by increasing  $b_0$ . Also note that  $\mathcal{Y}_{\lambda}$  is nonempty for all choices of  $\lambda$ . We will again implicitly assume that  $b_0$  is sufficiently large for Theorem 4.2 to hold.

The proof of Theorem 6.1 is based on a careful investigation of the boundary of  $\mathcal{Y}$  and the action of  $\mathcal{R}$  on this boundary. However, we need to introduce the set  $\mathcal{Y}_{\lambda}$  because we do not have a good enough lower bound on  $\varepsilon(\mathcal{R}f)$  for  $f \in \mathcal{Y}$ , see the discussion after Theorem 4.2.

**Definition 6.3.** A branch  $B$  of  $f^n$  is full if  $f^n$  maps  $B$  onto the domain of  $f$ ;  $B$  is trivial if  $f^n$  fixes both endpoints of  $B$ .

**Proposition 6.4.** *The boundary of  $\mathcal{Y}$  consists of three parts, namely  $f \in \partial\mathcal{Y}$  if and only if at least one of the following conditions hold:*

- (Y1) *the left or right branch of  $\mathcal{R}f$  is full or trivial,*
- (Y2)  $\varepsilon(f) = \varepsilon^-$  or  $\varepsilon(f) = \varepsilon^+$ , where  $\varepsilon(f) = 1 - c(f)$  and

$$\varepsilon^- = \min\{\varepsilon(g) \mid g \in \mathcal{K}\} \quad \text{and} \quad \varepsilon^+ = \max\{\varepsilon(g) \mid g \in \mathcal{K}\}$$

<sup>8</sup>The conjecture is that the restriction of  $\mathcal{R}$  to the set of infinitely renormalizable maps should contract maps of the same combinatorial type and this would imply uniqueness.



(Y3)  $\text{Dist } \phi(f) = \delta$  or  $\text{Dist } \psi(f) = \delta$  ( $\delta$  is the same as in Definition 4.1).  
Also, each condition occurs somewhere on  $\partial\mathcal{Y}$ .

Before giving the proof we need to introduce some new concepts and recall some established facts about families of Lorenz maps.

**Definition 6.5.** A slice (in the parameter plane) is any set of the form

$$\mathcal{S} = [0, 1]^2 \times \{c\} \times \{\phi\} \times \{\psi\},$$

where  $c$ ,  $\phi$  and  $\psi$  are fixed. We will permit ourselves to be a bit sloppy with notation and write  $(u, v) \in \mathcal{S}$  when it is clear which slice we are talking about (or if it is irrelevant).

A slice  $\mathcal{S} = [0, 1]^2 \times \{c\} \times \{\phi\} \times \{\psi\}$  induces a family of Lorenz maps

$$\mathcal{S} \ni (u, v) \mapsto f_{u,v} = (u, v, c, \phi, \psi) \in \mathcal{L}.$$

Any family induced from a slice is *full*, by which we mean that it realizes all possible combinatorics. See [Martens and de Melo, 2001] for a precise definition and a proof of this statement. For our discussion the only important fact is the following:

**Proposition 6.6.** *Let  $(u, v) \mapsto f_{u,v}$  be a family induced by a slice. Then this family intersects  $\mathcal{L}_{\bar{\omega}}$  for every  $\bar{\omega}$  such that  $\mathcal{L}_{\bar{\omega}} \neq \emptyset$ . Note that  $\bar{\omega}$  can be finite or infinite.*

*Proof.* This follows from [Martens and de Melo, 2001, Theorem A].  $\square$

Recall that  $C = \text{cl } L \cup R$  is the return interval for a renormalizable map, and the return times for  $L$  and  $R$  are  $a+1$  and  $b+1$ , respectively (see the end of Section 2).

**Lemma 6.7.** *Assume that  $f$  is renormalizable. Let  $(l, c)$  be the branch of  $f^{a+1}$  containing  $L$  and let  $(c, r)$  be the branch of  $f^{b+1}$  containing  $R$ . Then*

$$f^{a+1}(l) \leq l \quad \text{and} \quad f^{b+1}(r) \geq r.$$

*Proof.* This is a special case of [Martens and de Melo, 2001, Lemma 4.1].  $\square$

*Proof of Proposition 6.4.* Let us first consider the boundary of  $\mathcal{L}_{\omega}^0$ . If either branch of  $\mathcal{R}f$  is full or trivial, then we can perturb  $f$  in  $\mathcal{C}^0$  so that it no longer is renormalizable. Hence (Y1) holds on  $\partial\mathcal{L}_{\omega}^0$ . If  $f \in \mathcal{L}_{\omega}^0$  does not satisfy (Y1) then any sufficiently small  $\mathcal{C}^0$ -perturbation of  $f$  will still be renormalizable by Lemma 6.7. Hence the boundary of renormalization is exactly characterized by (Y1).

Conditions (Y2) and (Y3) are part of the boundary of  $\mathcal{K}$ . These boundaries intersect  $\mathcal{L}_{\omega}^S$  by Proposition 6.6 and hence these conditions are also boundary conditions for  $\mathcal{Y}$ .  $\square$

Fix  $1 - c_0 = \varepsilon_0 \in (\varepsilon^-, \varepsilon^+)$  and let  $\mathcal{S} = [0, 1]^2 \times \{c_0\} \times \{\text{id}\} \times \{\text{id}\}$ . Let  $\rho_t$  be the deformation retract onto  $\mathcal{S}$  defined by

$$\rho_t(u, v, c, \phi, \psi) = (u, v, c + t(c_0 - c), (1 - t)\phi, (1 - t)\psi), \quad t \in [0, 1].$$

In order to make sense of this formula it is important to note that the linear structure on the diffeomorphisms is that induced from  $\mathcal{C}^0$  via the nonlinearity operator  $N$  (see Remark 2.4). Hence, for example  $t\phi$  is by definition the diffeomorphism  $N^{-1}(tN\phi)$ . Let

$$\mathcal{R}_t = \rho_t \circ \mathcal{R}.$$

The choice of slice is somewhat arbitrary in what follows, except that we will have to be a little bit careful when choosing  $c_0$  as will be pointed out in the proof of the next lemma. However, it is important to note that the slice intersects  $\mathcal{Y}$ .

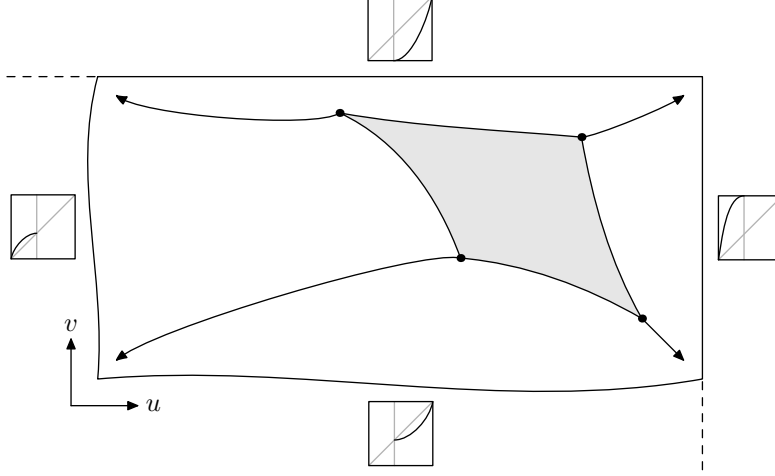


FIGURE 5. Illustration of the action of  $\rho_1 \circ \mathcal{R}|_{\mathcal{S}}$ . The shaded area corresponds to a full island. The boxes show what the branches of  $\rho_1 \circ \mathcal{R}f$  look like on each boundary piece.

**Lemma 6.8.** *There exists  $c_0$  such that  $\mathcal{R}_t$  has a fixed point on  $\partial\mathcal{Y}_\lambda$  for some  $t \in [0, 1]$  if and only if  $\mathcal{R}$  has a fixed point on  $\partial\mathcal{Y}_\lambda$ .*

*Remark 6.9.* The condition  $1 - c_1^+(\mathcal{R}f) \geq \lambda$  roughly states that  $v(\mathcal{R}f) \geq 1 - \lambda$ . Thus  $\mathcal{Y}_\lambda$  has another boundary condition given by  $c_1^+(\mathcal{R}f) = 1 - \lambda$ . Instead of treating this as separate boundary condition we subsume it into (Y1) by saying that the right branch is trivial also if  $c_1^+(\mathcal{R}f) = 1 - \lambda$ .

*Proof.* The ‘if’ statement is obvious since  $\mathcal{R} = \mathcal{R}_0$ , so assume that  $\mathcal{R}$  has no fixed point on  $\partial\mathcal{Y}_\lambda$ . Let  $f \in \partial\mathcal{Y}_\lambda$  and assume that  $\mathcal{R}_t f = f$  for some  $t > 0$ . We will show that this is impossible.

To start off choose  $\varepsilon_0 \in (\varepsilon^-, \varepsilon^+)$  and let  $c_0 = 1 - \varepsilon_0$  as usual (we will be more specific about the choice of  $\varepsilon_0$  later).

Note that (Y2) cannot hold for  $\mathcal{R}_t f$  since  $\varepsilon_0 \in (\varepsilon^-, \varepsilon^+)$  and hence the same is true for  $\varepsilon(\mathcal{R}_t f)$ , since  $t > 0$  and  $\varepsilon(\mathcal{R}f) \in [\varepsilon^-, \varepsilon^+]$  by Theorem 4.2.

Similarly, (Y3) cannot hold for  $\mathcal{R}_t f$  since the distortion of the diffeomorphic parts of  $\mathcal{R}f$  are not greater than  $\delta$  (by Theorem 4.2) and hence the distortion of the diffeomorphic parts of  $\mathcal{R}_t f$  are strictly smaller than  $\delta$  (since  $t > 0$ ).<sup>9</sup>

The only possibility is that  $f = \mathcal{R}_t f$  belongs to the boundary part described by condition (Y1).

If either branch of  $\mathcal{R}f$  is full then corresponding branch of  $\mathcal{R}_t f$  is full as well which shows that  $f$  cannot be fixed by  $\mathcal{R}_t$ , since a renormalizable map cannot have a full branch. Thus one of the branches of  $\mathcal{R}f$  must be trivial.

Assume that the left branch of  $\mathcal{R}f$  is trivial, that is  $c_1^-(\mathcal{R}f) = c(\mathcal{R}f)$ . In particular,  $\mathcal{R}f$  is not renormalizable since  $c_1^-$  for a renormalizable map is away from the critical point by Lemma 4.7. Because of this lemma we can assure that  $\mathcal{R}_s f$  is not renormalizable for all  $s \in [0, 1]$  by choosing  $\varepsilon_0$  close to  $\varepsilon^-$ . In particular,  $\mathcal{R}_t f$  is not renormalizable and hence cannot equal  $f$ .

<sup>9</sup> This follows from  $\text{Dist}(1 - t)\phi < \text{Dist } \phi$  if,  $t > 0$  and  $\text{Dist } \phi > 0$ .

Assume that the right branch of  $\mathcal{R}f$  is trivial (see Remark 6.9). Then we may without loss of generality assume that  $\lambda > \varepsilon^+$  and hence  $c_1^+(\mathcal{R}f) = 1 - \lambda$  (just choose  $\lambda$  not too small, or increase  $b_0$ ). In particular  $\mathcal{R}f$  is not renormalizable since that requires  $c_1^+(\mathcal{R}f)$  to be close to 0 by Lemma 4.7. The same holds for  $\mathcal{R}_s f$  for all  $s \in [0, 1]$  since  $\lambda > \varepsilon^+$ . In particular,  $f$  cannot be fixed by  $\mathcal{R}_t$  since  $f$  is renormalizable.

We have shown that  $f \notin \partial\mathcal{Y}_\lambda$  which is a contradiction and hence we conclude that  $\mathcal{R}_t f \neq f$  for all  $t \in [0, 1]$ .  $\square$

The slice  $\mathcal{S}$  intersects the set  $\mathcal{L}_\omega$  of renormalizable maps of type  $\omega$  by Proposition 6.6. This intersection can in general be a complicated set, but there will always be at least one connected component  $I$  of the interior such that the restricted family  $I \ni (u, v) \mapsto f_{u,v}$  is full [see Martens and de Melo, 2001, Theorem B]. Such a set  $I$  is called a full island. The action of  $\mathcal{R}$  on a full island is illustrated in Figure 5. Note that the action of  $\mathcal{R}$  on the boundary of  $I$  is given by (Y1) which also explains this figure.

**Lemma 6.10.** *Any extension of  $\mathcal{R}_1|_{\partial\mathcal{Y}_\lambda}$  to  $\mathcal{Y}_\lambda$  has a fixed point.*

*Proof.* If  $\mathcal{R}_1$  has a fixed point on  $\partial\mathcal{Y}_\lambda$  then there is nothing to prove, so assume that this is not the case.

Let  $\mathcal{S} = [0, 1]^2 \times \{c_0\} \times \{\text{id}\} \times \{\text{id}\}$ . By the above discussion there is a full island  $I \subset \mathcal{S}$ . Note that  $\partial I \subset \partial\mathcal{Y}_\lambda$ .

Pick any  $R : I \rightarrow \mathcal{S}$  such that  $R|_{\partial I} = \mathcal{R}_1|_{\partial I}$ . Now define the displacement map  $\delta : \partial I \rightarrow S^1$  by

$$\delta(x) = \frac{x - R(x)}{|x - R(x)|}.$$

This map is well-defined since  $\mathcal{R}_1$  was assumed not to have any fixed points on  $\partial\mathcal{Y}_\lambda$  and  $\partial I \subset \partial\mathcal{Y}_\lambda$ . The degree of  $\delta$  is nonzero since  $I$  is a full island. This implies that  $R$  has a fixed point in  $I$ , otherwise  $\delta$  would extend to all of  $I$  which would imply that the degree of  $\delta$  was zero. This finishes the proof since  $R$  was an arbitrary extension of  $\mathcal{R}_1|_{\partial I}$  and  $\partial I \subset \partial\mathcal{Y}_\lambda$ .  $\square$

**Proposition 6.11.**  *$\mathcal{R}_\omega$  has a fixed point.*

*Proof.* By the previous two lemmas either  $\mathcal{R}_\omega$  has a fixed point on  $\partial\mathcal{Y}_\lambda$  or we can apply Theorem A.1. In both cases  $\mathcal{R}_\omega$  has a fixed point.  $\square$

*Proof of Theorem 6.1.* Pick any sequence  $(\omega_0, \dots, \omega_{n-1})$  with  $\omega_i \in \Omega$ . The proof of the previous proposition can be repeated with

$$\mathcal{R}' = \mathcal{R}_{\omega_{n-1}} \circ \dots \circ \mathcal{R}_{\omega_0}$$

in place of  $\mathcal{R}$  to see that  $\mathcal{R}'$  has a fixed point  $f_*$ . But then  $f_*$  is a periodic point of  $\mathcal{R}$  and its combinatorial type is  $(\omega_0, \dots, \omega_{n-1})^\infty$ .  $\square$

## 7. DECOMPOSITIONS

In this section we introduce the notion of a decomposition. We show how to lift operators from diffeomorphisms to decompositions and also how decompositions can be composed in order to recover a diffeomorphism. This section is an adaptation of techniques introduced in Martens [1998].

**Definition 7.1.** A decomposition  $\bar{\phi} : T \rightarrow \mathcal{D}^2([0, 1])$  is an ordered sequence of diffeomorphisms labelled by a totally ordered and at most countable set  $T$ . Any such set  $T$  will be called a time set. The space  $\mathcal{D}$  is defined in Appendix B.

The space of decompositions  $\bar{\mathcal{D}}_T$  over  $T$  is the direct product

$$\bar{\mathcal{D}}_T = \prod_T \mathcal{D}^2([0, 1])$$

together with the  $\ell^1$ -norm

$$\|\bar{\phi}\| = \sum_{\tau \in T} \|\phi_\tau\|.$$

The notation here is  $\phi_\tau = \bar{\phi}(\tau)$ . The distortion of a decomposition is defined similarly:

$$\text{Dist } \bar{\phi} = \sum_{\tau \in T} \text{Dist } \phi_\tau.$$

The sum of two time sets  $T_0 \oplus T_1$  is the disjoint union

$$T_0 \oplus T_1 = \{(x, i) \mid x \in T_i, i = 0, 1\},$$

with order  $(x, i) < (y, i)$  if and only if  $x < y$ , and  $(x, 0) < (y, 1)$  for all  $x, y$ .

The sum of two decompositions

$$\bar{\phi}_0 \oplus \bar{\phi}_1 \in \bar{\mathcal{D}}_{T_0 \oplus T_1},$$

where  $\bar{\phi}_i \in \bar{\mathcal{D}}_{T_i}$ , is defined by  $\bar{\phi}_0 \oplus \bar{\phi}_1(x, i) = \bar{\phi}_i(x)$ . In other words,  $\bar{\phi}_0 \oplus \bar{\phi}_1$  is the diffeomorphisms of  $\bar{\phi}_0$  in the order of  $T_0$ , followed by the diffeomorphisms of  $\bar{\phi}_1$  in the order of  $T_1$ .

Note that  $\oplus$  is noncommutative on time sets as well as on decompositions.

*Remark 7.2.* Our approach to decompositions is somewhat different from that of Martens [1998]. In particular, we require a lot less structure on time sets and as such our definition is much more suitable to general combinatorics. Intuitively speaking, the structure that Martens [1998] puts on time sets is recovered from limits of the renormalization operator so we will also get this structure when looking at maps in the limit set of renormalization. We simply choose not to make it part of the definition to gain some flexibility.

**Proposition 7.3.** *The space of decompositions  $\bar{\mathcal{D}}_T$  is a Banach space.*

*Proof.* The nonlinearity operator takes  $\mathcal{D}^2([0, 1])$  bijectively to  $\mathcal{C}^0([0, 1]; \mathbb{R})$ . The latter is a Banach space so the same holds for  $\bar{\mathcal{D}}_T$ .  $\square$

**Definition 7.4.** Let  $T$  be a finite time set (i.e. of finite cardinality) so that we can label  $T = \{0, 1, \dots, n-1\}$  with the usual order of elements. The composition operator  $O : \bar{\mathcal{D}}_T \rightarrow \mathcal{D}^2$  is defined by

$$O\bar{\phi} = \phi_{n-1} \circ \dots \circ \phi_0.$$

The composition operator composes all maps in a decomposition in the order of  $T$ . We can also define partial composition operators

$$O_{[j, k]}\bar{\phi} = \phi_k \circ \dots \circ \phi_j, \quad 0 \leq j \leq k < n.$$

As a notational convenience we will write  $O_{\leq k}$  instead of  $O_{[0, k]}$  etc.

Next, we would like to extend the composition operator to countable time sets but unfortunately this is not possible in general. Instead of  $\mathcal{D}^2$  we will work with the space  $\mathcal{D}^3$  with the  $\mathcal{C}^1$ -nonlinearity norm:

$$\|\phi\|_1 = \|N\phi\|_{\mathcal{C}^1} = \max_{k=0,1} \{|D^k(N\phi)|\}, \quad \phi \in \mathcal{D}^3.$$

Define  $\bar{\mathcal{D}}_T^3 = \{\bar{\phi} : T \rightarrow \mathcal{D}^3 \mid \|\bar{\phi}\|_1 < \infty\}$ , where

$$\|\bar{\phi}\|_1 = \sum \|\phi_\tau\|_1.$$

Note that  $\|\cdot\|$  will still be used to denote the  $\mathcal{C}^0$ -nonlinearity norm.

**Proposition 7.5.** *The composition operator  $O : \bar{\mathcal{D}}_T^3 \rightarrow \mathcal{D}^2$  continuously extends to decompositions over countable time sets  $T$ .*

*Remark 7.6.* It is important to note that there is an inherent loss of smoothness when composing a decomposition over a countable time set. Starting with a bound on the  $\mathcal{C}^1$ -nonlinearity norm we only conclude a bound on the  $\mathcal{C}^0$ -nonlinearity norm of the composed map. This can be generalized; starting with a bound on the  $\mathcal{C}^{k+1}$ -nonlinearity norm, we can conclude a bound on the  $\mathcal{C}^k$ -nonlinearity norm for the composed map.

The reason why we loose one degree of smoothness is because we use the mean value theorem for one estimate in the Sandwich Lemma 7.9. If necessary it should be possible to replace this with for example a Hölder estimate which would lead to a slightly stronger statement.

In order to prove this proposition we will need the Sandwich Lemma which in itself relies on the following properties of the composition operator.

**Lemma 7.7.** *Let  $\bar{\phi} \in \bar{\mathcal{D}}_T$  be a decomposition over a finite time set  $T$ , and let  $\phi = O\bar{\phi}$ . Then*

$$e^{-\|\bar{\phi}\|} \leq |\phi'| \leq e^{\|\bar{\phi}\|}, \quad |\phi''| \leq \|\bar{\phi}\| e^{2\|\bar{\phi}\|}, \quad \text{and} \quad \|\phi\| \leq \|\bar{\phi}\| e^{\|\bar{\phi}\|}.$$

*If furthermore,  $\bar{\phi} \in \bar{\mathcal{D}}_T^3$ , then*

$$\|\phi\|_1 \leq (1 + \|\bar{\phi}\|) e^{2\|\bar{\phi}\|} \|\bar{\phi}\|_1.$$

*Remark 7.8.* Note that the lemma is stated for finite time sets, but the way we define the composition operator for countable time sets (see the proof of Proposition 7.5) will mean that the lemma also holds for countable time sets.

*Proof.* The bounds on  $|\phi'|$  and  $|\phi''|$  follow from an induction argument using only Lemma B.10.

Since  $T$  is finite we can label  $\bar{\phi}$  so that  $\phi = \phi_{n-1} \circ \dots \circ \phi_0$ . Let  $\psi_i = O_{<i}(\bar{\phi})$  and let  $\psi_0 = \text{id}$ . Now the bound on  $\|\phi\|$  follows from

$$N\phi(x) = \sum_{i=0}^{n-1} N\phi_i(\psi_i(x))\psi_i'(x),$$

which in itself is obtained from an induction argument using the chain rule for nonlinearities (see Lemma B.8).

Finally, take the derivative of the above equation to get

$$(N\phi)'(x) = \sum_{i=0}^{n-1} (N\phi_i)'(\psi_i(x))\psi_i'(x)^2 + N\phi_i(\psi_i(x))\psi_i''(x).$$

From this the bound on  $\|\phi\|_1$  follows.  $\square$

**Lemma 7.9** (Sandwich Lemma). *Let  $\phi = \phi_{n-1} \circ \dots \circ \phi_0$  and let  $\psi$  be obtained by “sandwiching  $\gamma$  inside  $\phi$ ,” that is,*

$$\psi = \phi_{n-1} \circ \dots \circ \phi_i \circ \gamma \circ \phi_{i-1} \circ \dots \circ \phi_0,$$

*for some  $i \in \{0, \dots, n\}$  (with the convention that  $\phi_n = \phi_{-1} = \text{id}$ ).*

*For every  $\lambda$  there exists  $K$  such that if  $\gamma, \phi_i \in \mathcal{D}^3$  and if  $\|\gamma\|_1 + \sum \|\phi_i\|_1 \leq \lambda$ , then  $\|\psi - \phi\| \leq K\|\gamma\|$ .*

*Proof.* Let  $\phi_+ = \phi_n \circ \dots \circ \phi_i$ , and let  $\phi_- = \phi_{i-1} \circ \dots \circ \phi_{-1}$ . Two applications of the chain rule for nonlinearities gives

$$\begin{aligned} |N\psi(x) - N\phi(x)| &= |N(\phi_+ \circ \gamma)(\phi_-(x)) - N\phi_+(\phi_-(x))| \cdot |\phi'_-(x)| \\ &= |N\phi_+(\gamma(y))\gamma'(y) - N\phi_+(y) + N\gamma(y)| \cdot |\phi'_-(x)|, \end{aligned}$$

where  $y = \phi_-(x)$ . By assumption  $N\phi_+ \in \mathcal{C}^1$  so by the mean value theorem there exists  $\eta \in [0, 1]$  such that

$$N\phi_+(\gamma(y)) = N\phi_+(y) + (N\phi_+)'(\eta) \cdot (\phi(y) - y).$$

Hence

$$\begin{aligned} |N\psi(x) - N\phi(x)| &\leq |\phi'_-(x)| \\ &\quad \cdot (|N\phi_+(y)| \cdot |\gamma'(y) - 1| + |\gamma'(y) \cdot (N\phi_+)'(\eta)| \cdot |\gamma(y) - y| + |N\gamma(y)|) \\ &\leq K_1 \cdot (K_2(e^{\|\gamma\|} - 1) + K_3(e^{2\|\gamma\|} - 1) + \|\gamma\|) \leq K\|\gamma\|. \end{aligned}$$

The constants  $K_i$  only depend on  $\lambda$  by Lemma 7.7. We have also used Lemma B.10 and Lemma B.11 in the penultimate inequality.  $\square$

*Proof of Proposition 7.5.* Let  $\bar{\phi} \in \bar{\mathcal{D}}_T^3$  and choose an enumeration  $\theta : \mathbb{N} \rightarrow T$ . Let  $\psi_n$  denote the composition of  $\{\phi_{\theta(0)}, \dots, \phi_{\theta(n-1)}\}$  in the order induced by  $T$ .

We claim that  $\{\psi_n\}$  is a Cauchy sequence in  $\mathcal{D}^2$ . Indeed, by applying the Sandwich Lemma with  $\lambda = \|\bar{\phi}\|_1$  we get a constant  $K$  only depending on  $\lambda$  such that:

$$\|\psi_n - \psi_m\| \leq \sum_{i=m}^{m+n-1} \|\psi_{i+1} - \psi_i\| \leq K \sum_{i=m}^{m+n-1} \|\phi_{\theta(i)}\| \rightarrow 0, \quad \text{as } m, n \rightarrow \infty.$$

Hence  $\phi = \lim \psi_n$  exists and  $\phi \in \mathcal{D}^2$ . This also shows that  $\phi$  is independent of the enumeration  $\theta$  and hence we can define  $O\bar{\phi} = \phi$ .  $\square$

We can now use the composition operator to lift operators from  $\mathcal{D}$  to  $\bar{\mathcal{D}}_T$ , starting with the zoom operators of Definition 2.9.

**Definition 7.10.** Let  $I \subset [0, 1]$  be an interval, let  $\bar{\phi} \in \bar{\mathcal{D}}_T^3$  and let  $I_\tau$  be the image of  $I$  under the diffeomorphism  $O_{<\tau}(\bar{\phi})$ . Define  $Z(\bar{\phi}; I) = \psi$ , where  $\psi_\tau = Z(\phi_\tau; I_\tau)$ , for every  $\tau \in T$ .

*Remark 7.11.* An equivalent way of defining the zoom operators on  $\bar{\mathcal{D}}_T^3$  is to let  $I_\tau = \psi_\tau^{-1}(J)$ , where  $\psi_\tau = O_{\geq\tau}(\bar{\phi})$ ,  $J = \phi(I)$ , and  $\phi = O\bar{\phi}(I)$ . This is equivalent since  $O\bar{\phi} = O_{\geq\tau}(\bar{\phi}) \circ O_{<\tau}(\bar{\phi})$ .

The original definition takes the view of zooming in on an interval in the domain of the decomposition, whereas the latter takes the view of zooming in on an interval in the range of the decomposition. We will make use of both of these points of view.

Zoom operators on diffeomorphisms are contractions for a fixed interval  $I$  by Lemma B.14. A similar statement holds for decompositions:

**Lemma 7.12.** *Let  $I \subset [0, 1]$  be an interval. If  $\bar{\phi} \in \bar{\mathcal{D}}_T^3$  then*

$$\|Z(\bar{\phi}; I)\| \leq e^{\|\bar{\phi}\|} \cdot \min\{|I|, |\phi(I)|\} \cdot \|\bar{\phi}\|,$$

where  $\phi = O\bar{\phi}$ .

*Remark 7.13.* Since we are only dealing with decompositions with very small norm this lemma is enough for our purposes. However, in more general situations the constant in front of  $\|\bar{\phi}\|$  may not be small enough. A way around this is to consider decompositions which compose to diffeomorphisms with negative Schwarzian derivative. Then all the intervals  $I_\tau$  will have hyperbolic lengths bounded by that of  $J$  (notation is as in Remark 7.11). This can then be used to show that zoom operators contract and the contraction can be bounded in terms of the hyperbolic length of  $J$ .

*Proof.* Using the notation of Definition 7.10 we have

$$\|Z(\bar{\phi}; I)\| = \sum_{\tau \in T} \|Z(\phi_\tau; I_\tau)\| \leq \sum_{\tau \in T} |I_\tau| \cdot \|\phi_\tau\| \leq \sup_{\tau \in T} |I_\tau| \cdot \|\bar{\phi}\|.$$

For every  $\tau$  there exists  $\xi_\tau \in I$  such that  $|I_\tau| = (O_{<\tau}(\bar{\phi}))'(\xi_\tau) \cdot |I|$  which together with Lemma 7.7 implies that  $|I_\tau| \leq e^{\|\bar{\phi}\|} \cdot |I|$ . Similarly, there exists  $\eta_\tau \in \phi(I)$  such that  $|\phi(I)| = (O_{\geq\tau}(\bar{\phi}))'(\eta_\tau) \cdot |I_\tau|$  so by Lemma 7.7  $|I_\tau| \leq e^{\|\bar{\phi}\|} \cdot |\phi(I)|$  as well.  $\square$

This contraction property of the zoom operators leads us to introduce the subspace of pure decompositions (the intuition is that renormalization contracts towards the pure subspace, see Proposition 8.8).

**Definition 7.14.** The subspace of pure decompositions  $\bar{\mathcal{Q}}_T \subset \bar{\mathcal{D}}_T$  consists of all decompositions  $\bar{\phi}$  such that  $\phi_\tau$  is a pure map for every  $\tau \in T$ .

The subspace of pure maps  $\mathcal{Q} \subset \mathcal{D}^\infty$  consists of restrictions of  $x^\alpha$  away from the critical point, that is

$$\mathcal{Q} = \{Z(x|x|^{\alpha-1}; I) \mid \text{int } I \not\ni 0\}.$$

A property of pure maps is that they can be parametrized by one real variable. We choose to parametrize the pure maps by their distortion with a sign and call this parameter  $s$ . The sign of  $s$  is positive for  $I$  to the right of 0 and negative for  $I$  to the left of 0. With this convention the graphs of pure maps will look like Figure 6.

*Remark 7.15.* Let  $\mu_s \in \mathcal{Q}$ . A calculation shows that

$$\text{Dist } \mu_s = |\log \mu'_s(1)/\mu'_s(0)|$$

and from this it is possible to deduce an expression for  $\mu_s$ :

$$(39) \quad \mu_s(x) = \frac{\left(1 + \left(\exp\left\{\frac{s}{\alpha-1}\right\} - 1\right)x\right)^\alpha - 1}{\exp\left\{\frac{\alpha s}{\alpha-1}\right\} - 1}, \quad x \in [0, 1], \quad s \neq 0,$$

and  $\mu_0 = \text{id}$ . We emphasize that the parametrization is chosen so that  $|s|$  equals the distortion of  $\mu_s$ . For this reason we call  $s$  the *signed* distortion of  $\mu_s$ . Figure 6 shows the graphs of  $\mu_s$  for different values of  $s$ . Equation (39) may at first seem to indicate that there is some sort of singular behavior at  $s = 0$  but this is not the case; the family  $s \mapsto \mu_s$  is smooth.

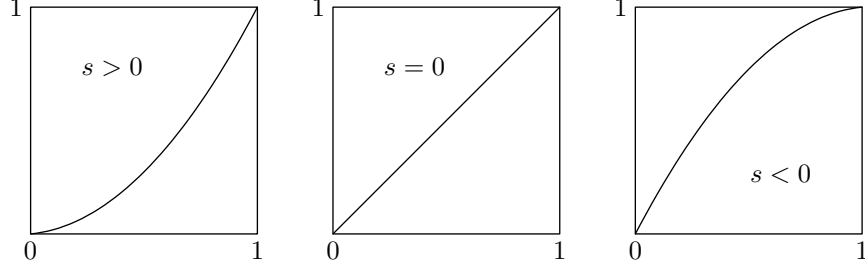


FIGURE 6. The graphs of a pure map  $\mu_s$  for different values of the signed distortion  $s$ .

The next two lemmas are needed in preparation for Proposition 8.8.

**Lemma 7.16.** *Let  $\phi \in \mathcal{D}^2$  and let  $I \subset [0, 1]$  be an interval. Then*

$$d(Z(\phi; I), \mathcal{Q}) \leq |I| \cdot d(\phi, \mathcal{Q}),$$

where the distance  $d(\cdot, \cdot)$  is induced by the  $\mathcal{C}^0$ -nonlinearity norm.

*Proof.* A calculation shows that

$$N\mu_s(x) = \frac{r_s(\alpha - 1)}{1 + r_s x}, \quad r_s = \exp\left\{\frac{s}{\alpha - 1}\right\} - 1.$$

Let  $I = [a, b]$  and let  $\zeta_I(x) = a + |I| \cdot x$ . Then

$$\begin{aligned} d(Z(\phi; I), \mathcal{Q}) &= \inf_{s \in \mathbb{R}} \max_{x \in [0, 1]} |N(Z(\phi; I))(x) - N\mu_s(x)| \\ &= \inf_{r > -1} \max_{x \in [0, 1]} \left| |I| \cdot N\phi(\zeta_I(x)) - \frac{r(\alpha - 1)}{1 + rx} \right| \\ &= \inf_{r > -1} \max_{x \in [0, 1]} \left| |I| \cdot N\phi(\zeta_I(x)) - \frac{r(\alpha - 1)}{1 + r(\zeta_I(x) - a)/|I|} \right| \\ &= |I| \cdot \inf_{\rho \notin [-\frac{1}{b}, -\frac{1}{a}]} \max_{x \in I} \left| N\phi(x) - \frac{\rho(\alpha - 1)}{1 + \rho x} \right|, \end{aligned}$$

where  $\rho = r/(b - (1 + r)a)$ . Note that  $1 + \rho x$  has a zero in  $[0, 1]$  if  $\rho \leq -1$ , so the infimum is assumed for  $\rho > -1$ . Thus

$$d(Z(\phi; I), \mathcal{Q}) = |I| \cdot \inf_{\rho > -1} \max_{x \in I} \left| N\phi(x) - \frac{\rho(\alpha - 1)}{1 + \rho x} \right|.$$

Taking the max over  $x \in [0, 1]$  finishes the proof.  $\square$

**Lemma 7.17.** *Let  $\bar{\phi} \in \bar{\mathcal{D}}_T^3$  and let  $I \subset [0, 1]$  be an interval. Then*

$$d(Z(\bar{\phi}; I), \bar{\mathcal{Q}}_T) \leq e^{\|\bar{\phi}\|} \cdot \min\{|I|, |\phi(I)|\} \cdot d(\bar{\phi}, \bar{\mathcal{Q}}_T),$$

where  $\phi = O\bar{\phi}$ .

*Proof.* Use Lemma 7.16 and a similar argument to that employed in the proof of Lemma 7.12.  $\square$

The pure decompositions have some very nice properties which we will make use of repeatedly.



**Proposition 7.18.** *If  $\bar{\phi} \in \bar{\mathcal{Q}}_T$  and  $\|\bar{\phi}\| < \infty$ , then  $\phi = O\bar{\phi}$  is in  $\mathcal{D}^\infty$  and  $\phi$  has nonpositive Schwarzian derivative.*

*Remark 7.19.* Note that  $\|\bar{\phi}\| < \infty$  is equivalent to  $\text{Dist } \bar{\phi} < \infty$ , since

$$\text{Dist } \mu = \int_0^1 |N\mu(x)| dx,$$

for pure maps  $\mu$ . Hence the norm bound can be replaced by a distortion bound and the above proposition still holds.

*Proof.* Let  $\eta$  be the nonlinearity of a pure map. A computation gives

$$D^k \eta(x) = \frac{(-1)^k k!}{(\alpha - 1)^k} \cdot \eta(x)^{k+1}.$$

Hence, if  $\eta$  is bounded then so are all of its derivatives (of course, the bound depends on  $k$ ). Thus Proposition 7.5 shows that  $\phi = O\bar{\phi}$  is well-defined and  $\phi \in \mathcal{D}^k$ , for all  $k \geq 2$  (use Remark 7.6).

Finally, every pure map has negative Schwarzian derivative so  $\phi$  must have non-positive Schwarzian derivative, since negative Schwarzian is preserved under composition by Lemma C.3.  $\square$

**Notation.** We put a bar over objects associated with decompositions to distinguish them from diffeomorphisms. Hence  $\bar{\phi}$  denotes a decomposition, whereas  $\phi$  denotes a diffeomorphism. Similarly,  $\bar{\mathcal{D}}$  denotes a set of decompositions, whereas  $\mathcal{D}$  is a set of diffeomorphisms.

Given a decomposition  $\bar{\phi} : T \rightarrow \mathcal{D}$ , we use the notation  $\phi_\tau$  to mean  $\bar{\phi}(\tau)$  and we call this the diffeomorphism at time  $\tau$ . Moreover, when talking about  $\bar{\phi}$  we consistently write  $\phi$  to denote the composed map  $O\bar{\phi}$ .

We will frequently consider the disjoint union of all decompositions instead of decompositions over some fixed time set  $T$  and for this reason we introduce the notation

$$\bar{\mathcal{D}} = \bigsqcup_T \bar{\mathcal{D}}_T \quad \text{and} \quad \bar{\mathcal{Q}} = \bigsqcup_T \bar{\mathcal{Q}}_T.$$

## 8. RENORMALIZATION OF DECOMPOSED MAPS

In this section we lift the renormalization operator to the space of decomposed Lorenz maps (i.e. Lorenz maps whose diffeomorphic parts are replaced with decompositions). We prove that renormalization contracts towards the subspace of pure decomposed maps. This will be used in later sections to compute the derivative of  $\mathcal{R}$  on its limit set.

**Definition 8.1.** Let  $T = (T_0, T_1)$  be a pair of time sets, and let  $\bar{\mathcal{D}}_T$  denote the product  $\bar{\mathcal{D}}_{T_0} \times \bar{\mathcal{D}}_{T_1}$ . The space of decomposed Lorenz maps  $\bar{\mathcal{L}}_T$  over  $T$  is the set  $[0, 1]^2 \times (0, 1) \times \bar{\mathcal{D}}_T$  together with structure induced from the Banach space  $\mathbb{R}^3 \times \bar{\mathcal{D}}_T$  with the max norm of the products.

**Definition 8.2.** The composition operator induces a map  $\bar{\mathcal{L}}_T^3 \rightarrow \mathcal{L}^2$  which (by slight abuse of notation) we will also denote  $O$ . Explicitly, if  $\bar{f} = (u, v, c, \bar{\phi}, \bar{\psi}) \in \bar{\mathcal{L}}_T$ , then  $f = O\bar{f}$  is defined by  $f = (u, v, c, O\bar{\phi}, O\bar{\psi})$ .

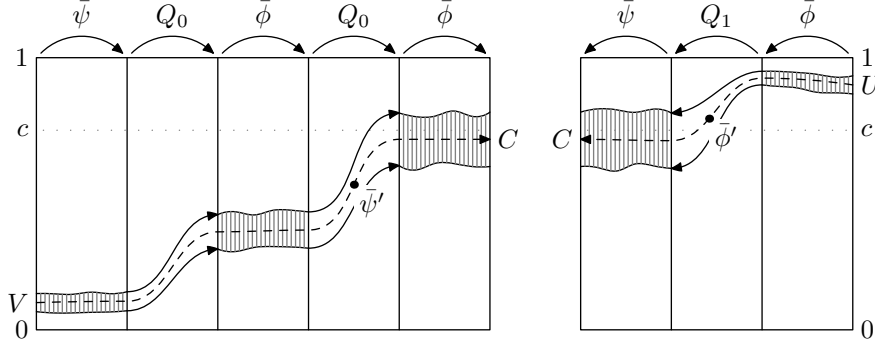


FIGURE 7. Illustration of the renormalization operator acting on decomposed Lorenz maps. First the decompositions are ‘glued’ to each other with  $Q$  according to the type of renormalization, here the type is  $(01, 100)$ . Then the interval  $C$  is pulled back, creating the shaded areas in the picture. The maps following the dashed arrows from  $U$  to  $C$  and from  $V$  to  $C$  represent the new decompositions before rescaling.

We will now define the renormalization operator on the space of decomposed Lorenz maps. Formally, the definition is identical to the definition of the renormalization operator on Lorenz maps. To illustrate this, let  $f = O\bar{f}$  be renormalizable. Then, by Lemma 2.11,  $\mathcal{R}f = (u', v', c', \phi', \psi')$ , where

$$(40) \quad u' = |Q(L)|/|U|, \quad v' = |Q(R)|/|V|, \quad c' = |L|/|C|,$$

$\phi' = Z(f^a \circ \phi; U)$  and  $\psi' = Z(f^b \circ \psi; V)$ . Zoom operators satisfy

$$Z(g \circ h; I) = Z(g; h(I)) \circ Z(h; I),$$

so we can write

$$\begin{aligned} \phi' &= Z(\psi; Q(U_a)) \circ Z(Q; U_a) \circ \cdots \circ Z(\psi; Q(U_1)) \circ Z(Q; U_1) \circ Z(\phi; U), \\ \psi' &= Z(\phi; Q(V_b)) \circ Z(Q; V_b) \circ \cdots \circ Z(\phi; Q(V_1)) \circ Z(Q; V_1) \circ Z(\psi; V). \end{aligned}$$

**Definition 8.3.** Define  $\mathcal{R}\bar{f} = (u', v', c', \bar{\phi}', \bar{\psi}')$ , where  $u', v', c'$  are given by (40) and

$$\begin{aligned} \bar{\phi}' &= Z(\bar{\phi}; U) \oplus Z(Q; U_1) \oplus Z(\bar{\psi}; Q(U_1)) \oplus \cdots \oplus Z(Q; U_a) \oplus Z(\bar{\psi}; Q(U_a)), \\ \bar{\psi}' &= Z(\bar{\psi}; V) \oplus Z(Q; V_1) \oplus Z(\bar{\phi}; Q(V_1)) \oplus \cdots \oplus Z(Q; V_b) \oplus Z(\bar{\phi}; Q(V_b)), \end{aligned}$$

where  $Z(Q; \cdot)$  is now interpreted as a decomposition over a singleton time set. See Figure 7 for an illustration of the action of  $\mathcal{R}$ .

**Definition 8.4.** The domain of  $\mathcal{R}$  on decomposed Lorenz maps is contained in the disjoint union  $\bar{\mathcal{L}} = \bigsqcup_T \bar{\mathcal{L}}_T$  over all time sets  $T$ . Just as before we let  $\bar{\mathcal{L}}_\omega$  denote all  $\omega$ -renormalizable maps in  $\bar{\mathcal{L}}$ ;  $\bar{\mathcal{L}}_{\bar{\omega}}$  denotes all maps in  $\bar{\mathcal{L}}$  such that  $\mathcal{R}^i \bar{f} \in \bar{\mathcal{L}}_{\omega_i}$ , where  $\bar{\omega} = (\omega_0, \omega_1, \dots)$ ; and  $\bar{\mathcal{L}}_\Omega = \bigcup_{\omega \in \Omega} \bar{\mathcal{L}}_\omega$ .

*Remark 8.5.* Note that  $\mathcal{R}$  takes the renormalizable maps of  $\bar{\mathcal{L}}_T$  into  $\bar{\mathcal{L}}_{T'}$ , where  $T' \neq T$  in general. This is the reason why we have to work with the disjoint union  $\bigsqcup_T \bar{\mathcal{L}}_T$ .

**Lemma 8.6.** *The composition operator is a semi-conjugacy. That is, the following square commutes*

$$\begin{array}{ccc} \bigcup \bar{\mathcal{L}}_\omega^3 & \xrightarrow{\mathcal{R}} & \bar{\mathcal{L}}^3 \\ o \downarrow & & \downarrow o \\ \bigcup \mathcal{L}_\omega^2 & \xrightarrow{\mathcal{R}} & \mathcal{L}^2 \end{array}$$

and  $O$  is surjective.

*Remark 8.7.* This lemma shows that we can use the composition operator to transfer results about decomposed Lorenz maps to Lorenz maps.

*Proof.* The square commutes by definition so let us focus on the surjectivity. Fix  $\tau \in T$  and define a map  $\Gamma_\tau : \mathcal{D} \rightarrow \bar{\mathcal{D}}_T$  by sending  $\phi \in \mathcal{D}$  to the decomposition  $\bar{\phi} : T \rightarrow \mathcal{D}$  defined by

$$\bar{\phi}(t) = \begin{cases} \phi, & \text{if } t = \tau, \\ \text{id}, & \text{otherwise.} \end{cases}$$

Then  $O \circ \Gamma_\tau = \text{id}$  which proves that  $O$  is surjective on  $\bar{\mathcal{D}}_T$  and hence it is also surjective on  $\bar{\mathcal{L}}_T$ .  $\square$

The main result for the renormalization operator on Lorenz maps was the existence of the invariant set  $\mathcal{K}$  for types in the set  $\Omega$ , see Section 4. It should come as no surprise that  $\mathcal{K}$  and  $\Omega$  will be central to our discussion on decomposed maps as well. The first result in this direction is the following.

**Proposition 8.8.** *If  $\bar{f} \in \bar{\mathcal{L}}_\omega^3$  is infinitely renormalizable with  $\bar{\omega} \in \Omega^\mathbb{N}$ , if  $\|\bar{\phi}\| \leq K$  and  $\|\bar{\psi}\| \leq K$ , and if  $O\bar{f} \in \mathcal{K} \cap \mathcal{L}^S$ , then the decompositions of  $\mathcal{R}^n \bar{f}$  are uniformly contracted towards the subset of pure decompositions.*

*Proof.* From the definition of the renormalization operator (and using the fact that  $d(Z(Q; I), \mathcal{Q}) = 0$ ) we get

$$d(\bar{\phi}', \bar{\mathcal{Q}}) = \sum_{i=1}^a d(Z(\bar{\psi}; Q(U_i)), \bar{\mathcal{Q}}) + d(Z(\bar{\phi}; U), \bar{\mathcal{Q}}).$$

Now apply Lemma 7.17 to get

$$d(\bar{\phi}', \bar{\mathcal{Q}}) \leq e^{\|\bar{\psi}\|} \sum_{i=2}^{a+1} |U_i| d(\bar{\psi}, \bar{\mathcal{Q}}) + e^{\|\bar{\phi}\|} |U_1| d(\bar{\phi}, \bar{\mathcal{Q}}).$$

From Section 4 we get that  $\sum |U_i|$  and  $\sum |V_i|$  may be chosen arbitrarily small (by choosing the return times sufficiently large). Now make these sums small compared with  $\max\{e^{\|\bar{\phi}\|}, e^{\|\bar{\psi}\|}\}$  to see that there exists  $\mu < 1$  (only depending on  $K$ ) such that

$$d(\bar{\phi}', \bar{\mathcal{Q}}) + d(\bar{\psi}', \bar{\mathcal{Q}}) \leq \mu [d(\bar{\phi}, \bar{\mathcal{Q}}) + d(\bar{\psi}, \bar{\mathcal{Q}})]. \quad \square$$

Our main goal is to understand the limit set of the renormalization operator and the above proposition will be central to this discussion.

**Definition 8.9.** The set of forward limits of  $\mathcal{R}$  restricted to types in  $\Omega$  is defined by

$$\mathcal{A}_\Omega = \bigcap_{n \geq 1} \mathcal{R}^n \left( \bigcup_{\bar{\omega} \in \Omega^n} \bar{\mathcal{L}}_{\bar{\omega}} \right).$$

*Remark 8.10.* In other words,  $\mathcal{A}_\Omega$  consists of all maps  $\bar{f}$  which have a complete past:

$$\bar{f} = \mathcal{R}_{\omega_{-1}} \bar{f}_{-1}, \quad \bar{f}_{-1} = \mathcal{R}_{\omega_{-2}} \bar{f}_{-2}, \quad \dots, \quad \omega_i \in \Omega.$$

This also describes how we can associate each  $\bar{f} \in \mathcal{A}_\Omega$  with a left infinite sequence  $(\dots, \omega_{-2}, \omega_{-1})$ .

**Proposition 8.11.**  $\mathcal{A}_\Omega$  is contained in the subset of pure decomposed Lorenz maps.

*Proof.* This is a direct consequence of Proposition 8.8.  $\square$

Since  $\mathcal{A}_\Omega$  is contained in the set of pure decomposed maps we will restrict our attention to this subset from now on. This is extremely convenient since pure decompositions satisfy some very strong properties, see Proposition 7.18, and it will allow us to compute the derivative at all points in  $\mathcal{A}_\Omega$  in Section 9.

Next we would like to lift the invariant set  $\mathcal{K}$  to the decomposed maps, but simply taking the preimage  $O^{-1}(\mathcal{K})$  will yield a set which is too large<sup>10</sup> so we will have to be a bit careful.

**Definition 8.12.** Let  $\delta$ ,  $\mathcal{K}$  and  $\Omega$  be the same as in Definition 4.1 and let

$$\varepsilon^- = \min\{\varepsilon(g) \mid g \in \mathcal{K}\}, \quad \varepsilon^+ = \max\{\varepsilon(g) \mid g \in \mathcal{K}\}.$$

Define

$$\bar{\mathcal{K}} = \{(u, v, c, \bar{\phi}, \bar{\psi}) \mid \varepsilon^- \leq 1 - c \leq \varepsilon^+, \text{Dist } \bar{\phi} \leq \delta, \text{Dist } \bar{\psi} \leq \delta, \bar{\phi}, \bar{\psi} \in \bar{\mathcal{Q}}\},$$

Note that  $\bar{\mathcal{K}}$  is defined analogously to  $\mathcal{K}$  but with the additional assumption that the decompositions are pure.

**Proposition 8.13.** If  $\bar{f} \in \bar{\mathcal{L}}_\Omega$  and  $1 - c_1^+(\mathcal{R}f) \geq \lambda > 0$  for some constant  $\lambda$  (not depending on  $b_0$ ), then

$$f \in \bar{\mathcal{K}} \implies \mathcal{R}f \in \bar{\mathcal{K}},$$

for  $b_0$  large enough.

*Proof.* Let  $f = O\bar{f} = (u, v, c, \phi, \psi)$ . Note first of all that  $\text{Dist } \bar{\phi} \leq \delta$  implies that  $\text{Dist } \phi \leq \delta$ , since  $\text{Dist}$  satisfies the subadditivity property

$$\text{Dist } \gamma_2 \circ \gamma_1 \leq \text{Dist } \gamma_1 + \text{Dist } \gamma_2.$$

Hence,  $f$  automatically satisfies the conditions of Theorem 4.2, so all we need to prove is that  $\text{Dist } \bar{\phi}' \leq \delta$  and  $\text{Dist } \bar{\psi}' \leq \delta$ . This is the reason why we define  $\bar{\mathcal{K}}$  by a distortion bound instead of a norm bound. Note that  $f$  has nonpositive Schwarzian since the decompositions are pure, see Proposition 7.18.

We will first show that the norm is invariant, then we transfer this invariance to the distortion. The reason why we consider the norm first is because it satisfies the contraction property in Lemma 7.12 which makes it easier to work with.

From the definition of  $\mathcal{R}$  and Lemma 7.12 we get

$$\begin{aligned} \|\bar{\phi}'\| &= \|Z(\bar{\phi}; U)\| + \sum_{i=1}^a \|Z(\bar{\psi}; Q(U_i))\| + \|Z(Q; U_i)\| \\ &\leq e^{\|\bar{\phi}\|} \|\bar{\phi}\| \cdot |U_1| + e^{\|\bar{\psi}\|} \|\bar{\psi}\| \sum_{i=2}^{a+1} |U_i| + \sum_{i=1}^a \|Z(Q; U_i)\|. \end{aligned}$$

<sup>10</sup>Any preimage under  $O$  contains decompositions whose norm is arbitrarily large. As an example of how things can go wrong, fix  $K > 0$  and consider  $\bar{\phi} : \mathbb{N} \rightarrow \mathcal{D}$  defined by  $\phi_{n+1} = \phi_n^{-1}$  and  $\|\phi_n\| = K$  for every  $n$ . Then  $\phi_{2n-1} \circ \dots \circ \phi_0 = \text{id}$  for every  $n$ , but  $\sum \|\phi_n\| = \infty$ .

The norm of a pure map is determined by how far away its domain is from the critical point. More precisely, we have that

$$\sum_{i=1}^a \|Z(Q; U_i)\| = (\alpha - 1) \sum_{i=1}^a \frac{|U_i|}{d(c, U_i)}.$$

Each term in this sum is bounded by the cross-ratio of  $U_i$  inside  $[c, 1]$ . Since maps with positive Schwarzian contract cross-ratio, since  $Sf < 0$ , and since  $U_i$  is a pull-back of  $C$  under an iterate of  $f$ , this cross-ratio is bounded by the cross-ratio  $\chi$  of  $C$  inside  $[c_1^+, 1]$ . Thus, the above sum is bounded by  $a(\alpha - 1)\chi$ . From the proof of Theorem 4.2 we know that  $\chi$  is of the order  $\varepsilon^t$  for some  $t > 0$ . Since  $a < b_0$  and  $b_0\varepsilon^t \rightarrow 0$  we see that the above sum has a uniform bound which tends to zero as  $b_0 \rightarrow \infty$ .

A similar argument for  $\bar{\psi}'$  gives

$$\begin{aligned} \|\bar{\phi}'\| + \|\bar{\psi}'\| &\leq (\|\bar{\phi}\| + \|\bar{\psi}\|) \exp\{\|\bar{\phi}\| + \|\bar{\psi}\|\} \left( \sum |U_i| + \sum |V_i| \right) + m \\ &= k (\|\bar{\phi}\| + \|\bar{\psi}\|) + m, \end{aligned}$$

where  $m = \sum \|Z(Q; U_i)\| + \sum \|Z(Q; V_i)\|$ . Hence

$$\|\bar{\phi}\| + \|\bar{\psi}\| \leq \delta \implies \|\bar{\phi}'\| + \|\bar{\psi}'\| \leq \delta, \quad \text{if } \delta \geq m/(1 - k).$$

By Definition 4.1,  $\delta = (1/b_0)^2$  and  $\varepsilon$  is of the order  $\alpha^{-b_0 K}$ , and by the above  $m$  is of the order  $b_0\varepsilon^t$ . Hence  $\delta \geq m/(1 - k)$  for  $b_0$  large enough.

The final observation which we use to finish the proof is that if  $\gamma \in \mathcal{Q}$  then

$$\|\gamma\| = (\alpha - 1) \cdot \left( \exp \left\{ \frac{\text{Dist } \gamma}{\alpha - 1} \right\} - 1 \right).$$

That is  $\|\gamma\| \approx \text{Dist } \gamma$  for pure maps  $\gamma$  with small distortion. This allows us to slightly modify the above invariance argument for the norm so that it holds for the distortion as well.  $\square$

## 9. THE DERIVATIVE

The tangent space of  $\mathcal{R}$  on the pure decomposed Lorenz maps can be written  $X \times Y$ , where  $X = \mathbb{R}^2$  and  $Y = \mathbb{R} \times \ell^1 \times \ell^1$ . The coordinates on  $X$  correspond to the  $(u, v)$  coordinates on  $\tilde{\mathcal{L}}_T$ . Let  $(x, y) \in X \times Y$  denote the coordinates on the tangent space and recall that we are using the max norm on the products. The derivative of  $\mathcal{R}$  at  $\bar{f}$  is denoted

$$(41) \quad D\mathcal{R}_{\bar{f}} = M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where  $M_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $M_2 : Y \rightarrow \mathbb{R}^2$ ,  $M_3 : \mathbb{R}^2 \rightarrow Y$  and  $M_4 : Y \rightarrow Y$  are bounded linear operators.

Note that the differentiability of  $\mathcal{R}$  follows from the calculations in this section since they could be carried out to the second order. However, we have chosen to only make first order calculations since they are already quite involved.

*Remark 9.1.* The fact that the derivative on the pure decomposed maps can be written as an infinite matrix is one of the reasons why we restrict ourselves to the pure decompositions. Deformations of pure decompositions are also easy to deal with since they are ‘monotone’ in the sense that the dynamical intervals that define

the renormalization move monotonically under such deformations. This makes it possible to estimate the elements of the derivative matrix.

**Theorem 9.2.** *There exist constants  $k$  and  $K$  such that if  $\bar{f} \in \bar{\mathcal{K}} \cap \bar{\mathcal{L}}_\Omega$  and  $1 - c_1^+(\mathcal{R}\bar{f}) \geq \lambda$  for some  $\lambda \in (0, 1)$  (not depending on  $\bar{f}$ ), then*

$$\begin{aligned} \|M_1 x\| &\geq k \min\{|U|^{-1}, |V|^{-1}\} \cdot \|x\|, & \|M_2\| &\leq K|C|^{-1}, \\ \|M_3 x\| &\leq K\rho' \left( \frac{|x_1|}{|U|} + \frac{|x_2|}{|V|} \right), & \|M_4\| &\leq K\rho'|C|^{-1}, \end{aligned}$$

where  $\rho' = \max\{\varepsilon', \text{Dist } \bar{\phi}', \text{Dist } \bar{\psi}'\}$  and  $b_0$  is sufficiently large.

*Remark 9.3.* The set  $\bar{\mathcal{K}}$  is introduced in Definition 8.12 and  $\Omega$  is given by Definition 4.1 as always. Note that the decompositions of  $\bar{f} \in \bar{\mathcal{K}}$  are pure and hence  $O\bar{f} \in \mathcal{L}^S$  by Proposition 7.18. Finally, the condition on  $c_1^+(\mathcal{R}\bar{f})$  is used to avoid maps whose renormalization has a right branch which is close to being trivial (see the proof of Proposition 9.12).

*Proof.* The proof of this theorem is split up into a few propositions that are in this section. The estimate for  $M_1$  is given in Corollary 9.11. The estimates for  $M_2$  and  $M_4$  follow from Propositions 9.12 and 9.15. Finally, the estimate for  $M_3$  follows from Propositions 9.12 and 9.13.  $\square$

**Notation.** Let  $\bar{f} = (u, v, c, \bar{\phi}, \bar{\psi})$  and as always use primes to denote the renormalization  $\mathcal{R}\bar{f} = (u', v', c', \bar{\phi}', \bar{\psi}')$ . We introduce special notation for the diffeomorphic parts of the renormalization *before* rescaling:

$$(42) \quad \Phi = f_1^a \circ \phi, \quad \Psi = f_0^b \circ \psi,$$

so that  $\Phi : U \rightarrow C$ ,  $\Psi : V \rightarrow C$ , and  $C = (p, q)$ . Note that  $p$  and  $q$  are by definition periodic points of periods  $a + 1$  and  $b + 1$ , respectively.

We will use the notation  $\partial_s t$  to denote the partial derivative of  $t$  with respect to  $s$ . In the formulas below we write  $\partial t$  to mean the partial derivative of  $t$  with respect to any direction.

The notation  $g(x) \asymp y$  is used to mean that there exists  $K < \infty$  not depending on  $g$  such that  $K^{-1}y \leq g(x) \leq Ky$  for all  $x$  in the domain of  $g$ .

The  $\partial$  operator satisfies the following rules:

**Lemma 9.4.** *The following expressions hold whenever they make sense:*

$$(43) \quad \partial(f \circ g)(x) = \partial f(g(x)) + f'(g(x))\partial g(x),$$

$$(44) \quad \partial(f^{n+1})(x) = \sum_{i=0}^n Df^{n-i}(f^{i+1}(x))\partial f(f^i(x)),$$

$$(45) \quad \partial(f^{-1})(x) = -\frac{\partial f(f^{-1}(x))}{f'(f^{-1}(x))}.$$

Furthermore, if  $f(p) = p$  then

$$(46) \quad \partial p = -\frac{\partial f(p)}{f'(p) - 1}.$$

*Remark 9.5.* The  $\partial$  operator clearly also satisfies the product rule

$$(47) \quad \partial(f \cdot g)(x) = \partial f(x)g(x) + f(x)\partial g(x).$$

This and the chain rule gives the quotient rule

$$(48) \quad \partial(f/g)(x) = \frac{\partial f(x)g(x) - f(x)\partial g(x)}{g(x)^2}.$$

*Proof.* Equation (43) implies the other three. The second equation is an induction argument and the last two follow from

$$0 = \partial(x) = \partial(f \circ f^{-1}(x)) = \partial f(f^{-1}(x)) + f'(f^{-1}(x))\partial(f^{-1}(x)),$$

and

$$\partial(p) = \partial(f(p)) = \partial f(p) + f'(p)\partial p.$$

Equation (43) itself can be proved by writing  $f_\varepsilon(x) = f(x) + \varepsilon \hat{f}(x)$ ,  $g_\varepsilon(x) = g(x) + \varepsilon \hat{g}(x)$  and using Taylor expansion:

$$\begin{aligned} f_\varepsilon(g_\varepsilon(x)) &= f_\varepsilon(g(x)) + \varepsilon f'_\varepsilon(g(x))\hat{g}(x) + \mathcal{O}(\varepsilon^2) \\ &= f(g(x)) + \varepsilon \{ \hat{f}(g(x)) + f'(g(x))\hat{g}(x) \} + \mathcal{O}(\varepsilon^2). \end{aligned} \quad \square$$

We now turn to computing the derivative matrix  $M$ . The first three rows of  $M$  are given by the following formulas.

**Lemma 9.6.** *The partial derivatives of  $u'$ ,  $v'$  and  $c'$  are given by*

$$\begin{aligned} \partial u' &= \frac{\partial(Q_0(c) - Q_0(p)) - u' \cdot \partial(\Phi^{-1}(q) - \Phi^{-1}(p))}{|U|}, \\ \partial v' &= \frac{\partial(Q_1(q) - Q_1(c)) - v' \cdot \partial(\Psi^{-1}(q) - \Psi^{-1}(p))}{|V|}, \\ \partial c' &= \frac{\partial(c - p) - c' \cdot \partial(q - p)}{|C|}. \end{aligned}$$

*Proof.* Use (40), Lemma 9.4 and Remark 9.5.  $\square$

Let us first consider how to use these formulas when deforming in the  $u$ ,  $v$  or  $c$  directions (i.e. the first three columns of  $M$ ). Almost everything in these formulas is completely explicit — we have expressions for  $Q_0$  and  $Q_1$  so evaluating for example  $\partial_u Q_0(c)$  is routine. In order to evaluate for example the term  $\partial_u \Psi^{-1}(q)$  we make use of (45) and (44). This involves estimating the sum in (44) which can be done with mean value theorem estimates. The terms  $\partial p$  and  $\partial q$  are evaluated using (46) and the fact that  $p = \Phi \circ Q_0(p)$  and  $q = \Psi \circ Q_1(q)$ . There are a few shortcuts to make the calculations simpler as well, for example  $\partial_u \Phi = 0$  since  $\Phi$  does not contain  $Q_0$  which is the only term that depends on  $u$ , and so on.

Deforming in the  $\bar{\phi}$  or  $\bar{\psi}$  directions (there are countably many such directions) is similar. Here we make use of the fact that the decompositions are pure and we have an explicit formula (39) for pure maps where the free parameter represents the signed distortion (see Remark 7.15), so we can compute their derivative, partial derivative with respect to distortion etc. These deformations will affect the partial derivatives of any expression involving  $\Phi$  or  $\Psi$ , but all others will not ‘see’ these deformations. The calculations involved do not make any particular use of which direction we deform in, so even though there are countably many directions we essentially only need to perform one calculation for  $\bar{\phi}$  and another for  $\bar{\psi}$ .

We now turn to computing the partial derivatives of  $\bar{\phi}'$  and  $\bar{\psi}'$ .

**Lemma 9.7.** *Let  $\mu_{s'} = Z(\mu_s; I)$ , where  $\mu_s, \mu_{s'} \in \mathcal{Q}$  and  $I = [x, y]$ . Then*

$$\partial s' = N\mu_s(y)\partial y - N\mu_s(x)\partial x + \frac{\partial(D\mu_s)(y)}{D\mu_s(y)} - \frac{\partial(D\mu_s)(x)}{D\mu_s(x)}.$$

*Proof.* By definition  $s = \log\{D\mu_s(1)/D\mu_s(0)\}$ . Distortion is invariant under zooming, so this shows that  $s' = \log\{D\mu_s(y)/D\mu_s(x)\}$ . A calculation gives

$$\partial(\log D\mu_s(x)) = \frac{\partial(D\mu_s)(x)}{D\mu_s(x)} + N\mu_s(x)\partial x. \quad \square$$

By definition  $\bar{\phi}'$  consists of maps of the form  $Z(\mu_s; I)$  (as well as finitely many of the form  $Z(Q; I)$  but these can be thought of as  $\lim_{s \rightarrow \pm\infty} Z(\mu_s; I)$ ). Hence the above lemma shows us how to compute the partial derivatives at each time in  $\bar{\phi}'$ . Note that we implicitly identify  $\mathbb{R}$  with  $\mathcal{Q}$  via  $s \mapsto \mu_s$ .

In order to use the lemma we also need a way to evaluate the terms  $\partial x$  and  $\partial y$ . One way to do this is to express these in terms of  $\partial p$  and  $\partial q$  which have already been computed at this stage. If we let  $T : I \rightarrow [p, q]$  denote the ‘transfer map’ to  $C$ , then  $p = T(x)$  and hence (43) shows that

$$\partial x = \frac{\partial p - \partial T(x)}{DT(x)}.$$

The terms  $\partial T$  and  $DT$  can be bounded by  $\partial\Phi$  and  $D\Phi$  (or  $\partial\Psi$  and  $D\Psi$ ) all of which have already been computed as well.

We will now compute the  $M_1$  part of the derivative matrix. Note that  $M_1$  has nothing to do with decompositions so the following proposition is stated for nondecomposed Lorenz maps.

**Proposition 9.8.** *If  $f \in \mathcal{K} \cap \mathcal{L}_\Omega^S$ , then*

$$M_1 = \begin{pmatrix} \frac{1}{|U|} \left( 1 + \frac{1-u'}{u} \frac{Q(p)}{Df^{a+1}(p)-1} \right) & -\frac{1}{|U|} \frac{u'}{v} \frac{D\Psi(\Psi^{-1}(q))}{D\Phi(\Phi^{-1}(q))} \frac{1-Q(q)}{Df^{b+1}(q)-1} \\ -\frac{1}{|V|} \frac{v'}{u} \frac{D\Phi(\Phi^{-1}(p))}{D\Psi(\Psi^{-1}(p))} \frac{Q(p)}{Df^{a+1}(p)-1} & \frac{1}{|V|} \left( 1 + \frac{1-v'}{v} \frac{1-Q(q)}{Df^{b+1}(q)-1} \right) \end{pmatrix} + M_1^e,$$

where the error term  $M_1^e$  is negligible.

*Remark 9.9.* From Section 4 we know that the critical point of the renormalization is very close to 1 and that the distortion of the diffeomorphic parts of the renormalization are bounded by  $\delta$  (which is very small). From these two facts we can get an idea of the size of the entries of  $M_1$ . For example,  $u'$  is very close to 1 since  $c'$  is (and  $\mathcal{R}f$  is assumed to be nontrivial so  $u' \geq e^{-\delta}c'$ ). Furthermore,  $Df^{a+1}(p) = D(\mathcal{R}f)(0)$  and  $Df^{b+1} = D(\mathcal{R}f)(1)$  since an affine change of coordinates does not change the derivative, so the distortion bounds for  $\mathcal{R}f$  implies that  $Df^{a+1}(p) \asymp \alpha u'/c'$  and  $Df^{b+1}(q) \asymp \alpha v'/\varepsilon'$  (these expressions come from the derivative of  $Q(x)$ , see (1)).

*Proof.* We begin by computing  $\partial p$  and  $\partial q$ . Use  $\Phi \circ Q_0(p) = p$ ,  $\Psi \circ Q_1(q) = q$ , and (46) to get

$$(49) \quad \partial_u p = -\frac{D\Phi(Q_0(p))\partial_u Q_0(p)}{Df^{a+1}(p)-1}, \quad \partial_u q = -\frac{\partial_u \Psi(Q_1(q))}{Df^{b+1}(q)-1},$$

$$(50) \quad \partial_v p = -\frac{\partial_v \Phi(Q_0(p))}{Df^{a+1}(p)-1}, \quad \partial_v q = -\frac{D\Psi(Q_1(q))\partial_v Q_1(q)}{Df^{b+1}(q)-1}.$$

Here we have used that  $\partial_u \Phi = 0$  and  $\partial_v \Psi = 0$ .



Next, let us estimate  $\partial_u \Psi$ . Let  $x \in V$  and let  $x_i = f^i \circ \psi(x)$ . From (44) we get

$$\partial_u \Psi(x) = \partial_u (f^b \circ \psi)(x) = \partial_u f(x_{b-1}) + \sum_{i=1}^{b-1} Df^{b-i}(x_i) \partial_u f(x_{i-1}),$$

where  $\partial_u f(x) = \phi'(Q_0(x))Q_0(x)/u$ . Note that  $\partial_u f(x_{i-1}) \leq e^{2\delta} x_i/u$ . In order to bound the sum we divide the estimate into two parts. Let  $n < b$  be the smallest integer such that  $Df(x_i) \leq 1$  for all  $i \geq n$ . In the part where  $i < n$  we estimate

$$Df^{b-i}(x_i)x_i = Df^{n-i}(x_i)Df^{b-n}(x_n)x_i \leq K_1 \frac{x_n}{x_i} Df(x_{b-1})x_i \leq K_2 \varepsilon^{1-1/\alpha}.$$

Here we have used the mean value theorem to find  $\xi_i \leq x_i$  such that  $Df^{n-i}(\xi_i) = x_n/x_i$  and  $Df^{n-i}(x_i) \leq K_1 Df^{n-i}(\xi_i)$ , since  $\phi$  has very small distortion. In the part where  $i \geq n$  we estimate

$$Df^{b-i}(x_i)x_i \leq Df(x_{b-1})x_i \leq K \varepsilon^{1-1/\alpha}.$$

Summing over the two parts gives us the estimate

$$\sum_{i=1}^{b-1} Df^{b-i}(x_i) \partial_u f(x_{i-1}) \leq K(b-1) \varepsilon^{1-1/\alpha}.$$

Hence

$$(51) \quad \partial_u \Psi(x) = \partial_u f(f^{b-1} \circ \psi(x)) + \mathcal{O}(b \varepsilon^{1-1/\alpha}) \approx 1.$$

We will now estimate  $\partial_v \Phi$ . Let  $x \in U$  and let  $x_i = f^i \circ \phi(x)$ . Similarly to the above, we have

$$\partial_v \Phi(x) = \partial_v f(x_{a-1}) + \sum_{i=1}^{a-1} Df^{a-i}(x_i) \partial_v f(x_{i-1}),$$

where  $\partial_v f(x) = -\psi'(Q_1(x))(1 - Q_1(x))/v$ . By the mean value theorem there exists  $\xi_i \in [x_i, 1]$  such that  $Df^{a-i}(\xi_i) = (1 - x_a)/(1 - x_i)$ , since  $f^{a-i}(x_i) = x_a$ . From Lemma 4.9 it follows that  $Df^{a-i}(x_i) \asymp Df^{a-i}(\xi_i)$ . Putting all of this together we get that the sum above is proportional to

$$\sum_{i=1}^{a-1} Df^{a-i}(\xi_i)(1 - x_i) = (a-1)(1 - x_a).$$

Thus

$$(52) \quad \partial_v \Phi(x) \asymp -a\varepsilon,$$

since  $x_a \in C$  and hence  $1 - x_a = \varepsilon + \mathcal{O}(|C|) \approx \varepsilon$ .

We now have all the ingredients we need to compute  $M_1$ . Lemma 9.6 shows that

$$\begin{aligned} |U| \partial_u u' &= \partial_u Q_0(c) - \partial_u Q_0(p) - Q'_0(p) \partial_u p \\ &\quad - u' (D\Phi^{-1}(q) \partial_u q - D\Phi^{-1}(p) \partial_u p). \end{aligned}$$

Here we have used  $\partial_u \Phi = 0$ . Now use (49) to get

$$Q'_0(p) \partial_u p = -\partial_u Q_0(p) \frac{Df^{a+1}(p)}{Df^{a+1}(p) - 1}, \quad D\Phi^{-1}(p) \partial_u p = -\frac{\partial_u Q_0(p)}{Df^{a+1}(p) - 1}.$$

Thus

$$(53) \quad |U| \partial_u u' = 1 + \frac{(1 - u') \partial_u Q_0(p)}{Df^{a+1}(p) - 1} + \frac{u' \partial_u \Psi(Q_1(q))}{D\Phi(\Phi^{-1}(q))(Df^{b+1}(q) - 1)}.$$

The last term is much smaller than one because of (51) and since  $|D\Phi| \gg 1$  (and also  $Df^{b+1}(q) \asymp v'\alpha/\varepsilon' \geq e^{-\delta}\alpha$ ).

From Lemma 9.6 we get

$$|V|\partial_v v' = \partial_v Q_1(q) + Q'_1(q)\partial_v q - \partial_v Q_1(c) \\ - v'(D\Psi^{-1}(q)\partial_v q - D\Psi^{-1}(p)\partial_v p).$$

Here we have used  $\partial_v \Psi = 0$ . Now use (50) to get

$$Q'_1(q)\partial_v q = -\frac{\partial_v Q_1(q)Df^{b+1}(q)}{Df^{b+1}(q) - 1}, \quad D\Psi^{-1}(q)\partial_v q = -\frac{\partial_v Q_1(q)}{Df^{b+1}(q) - 1}.$$

Thus

$$(54) \quad |V|\partial_v v' = 1 - \frac{(1-v')\partial_v Q_1(q)}{Df^{b+1}(q) - 1} - \frac{v'\partial_v \Phi(Q_0(p))}{D\Psi(\Psi^{-1}(p))(Df^{a+1}(p) - 1)}.$$

The last term is much smaller than one by (52) and since  $|D\Psi| \gg 1$  (and also  $Df^{a+1}(p) \asymp \alpha u'/c' \geq e^{-\delta}\alpha$ ).

From Lemma 9.6 we get

$$|U|\partial_v u' = -Q'_0(p)\partial_v p - u' \left( \partial_v \Phi^{-1}(q) + D\Phi^{-1}(q)\partial_v q \right. \\ \left. - \partial_v \Phi^{-1}(p) - D\Phi^{-1}(p)\partial_v p \right).$$

Let us prove that that the dominating term is the one with  $\partial_v q$ . From (50) we get

$$\partial_v q = -\frac{\partial_v Q_1(q)}{Q'_1(q)} \frac{Df^{b+1}(q)}{Df^{b+1}(q) - 1},$$

which diverges as  $b_0 \rightarrow \infty$ , since  $|R|/\varepsilon \rightarrow 0$  and hence  $Q'_1(q) \rightarrow 0$  (by the proof of Proposition 4.10). From (50) and (52) we get that  $\partial_v p \rightarrow 0$ , which shows that the last term is dominated by the term with  $\partial_v q$ . Now,  $\partial_v \Phi^{-1}(x) = -\partial_v \Phi(x)/D\Phi(x)$ , which combined with (52) shows that the term with  $\partial_v q$  dominates the two terms with  $\partial_v \Phi^{-1}$ . Furthermore

$$Q'_0(p)\partial_v p = -\frac{\partial_v \Phi(Q_0(p))}{D\Phi(Q_0(p))} \frac{Df^{a+1}(p)}{Df^{a+1}(p) - 1},$$

which combined with (52) shows that the term with  $\partial_v q$  dominates the above term. Thus

$$(55) \quad |U|\partial_v u' = u' \frac{D\Psi(\Psi^{-1}(q))}{D\Phi(\Phi^{-1}(q))} \frac{\partial_v Q_1(q)}{Df^{b+1}(q) - 1} + e,$$

where the error term  $e$  is tiny compared with the other term on the right-hand side.

From Lemma 9.6 we get

$$|V|\partial_u v' = Q'_1(q)\partial_u q - v' \left( \partial_u \Psi^{-1}(q) + D\Psi^{-1}(q)\partial_u q \right. \\ \left. - \partial_u \Psi^{-1}(p) - D\Psi^{-1}(p)\partial_u p \right).$$

Let us prove that that the dominating term is the one with  $\partial_u p$ . From (49) we get

$$\partial_u p = -\frac{\partial_u Q_0(p)}{Q'_0(p)} \frac{Df^{a+1}(p)}{Df^{a+1}(p) - 1},$$

which diverges as  $b_0 \rightarrow \infty$ , since  $|L|/c \rightarrow 0$  and hence  $Q'_0(p) \rightarrow 0$ . From (49) and (51) we get that  $\partial_u q$  is bounded and hence the  $\partial_u p$  term dominates the second term involving  $\partial_u q$ . Now,  $\partial_u \Psi^{-1}(x) = -\partial_u \Psi(y)/D\Psi(y)$ ,  $y = \Psi^{-1}(x)$ , which combined with (51) shows that the  $\partial_u p$  term dominates the two terms involving  $\partial_u \Psi^{-1}$ . Furthermore

$$Q'_1(q)\partial_u q = -\frac{\partial_u \Psi(Q_1(q))}{D\Psi(Q_1(q))} \frac{Df^{b+1}(q)}{Df^{b+1}(q) - 1},$$

which combined with (51) shows that the  $\partial_u p$  term dominates the above term. Thus

$$(56) \quad |V|\partial_u v' = -v' \frac{D\Phi(\Phi^{-1}(p))}{D\Psi(\Psi^{-1}(p))} \frac{\partial_u Q_0(p)}{Df^{a+1}(p) - 1} + e,$$

where the error term  $e$  is tiny compared with the other term on the right-hand side.  $\square$

**Corollary 9.10.** *If  $f \in \mathcal{K} \cap \mathcal{L}_\Omega^S$ , then  $\det M_1 > 0$  for  $b_0$  large enough.*

*Proof.* From Proposition 4.11 we get that

$$\frac{D\Phi(\Phi^{-1}(p))}{D\Phi(\Phi^{-1}(q))} \frac{D\Psi(\Psi^{-1}(q))}{D\Psi(\Psi^{-1}(p))} \leq e^{2\delta},$$

since distortion is invariant under linear rescaling. Now use this together with Proposition 9.8 to get

$$|U||V|\det M_1 > 1 - e^{2\delta} \frac{u'v'}{uv} \frac{Q(p)(1-Q(q))}{(Df^{a+1}(p) - 1)(Df^{b+1}(q) - 1)}.$$

Equation (1) gives

$$\frac{Q(p)}{u} = 1 - \left(\frac{|L|}{c}\right)^\alpha < 1 \quad \text{and} \quad \frac{1-Q(q)}{v} = 1 - \left(\frac{|R|}{\varepsilon}\right)^\alpha < 1.$$

Remark 9.9 allows us to estimate

$$\frac{u'}{Df^{a+1}(p) - 1} \leq \frac{e^\delta}{\alpha - e^{2\delta}} \quad \text{and} \quad \frac{v'}{Df^{b+1}(q) - 1} \leq \varepsilon' \frac{e^\delta}{\alpha - e^{2\delta}}.$$

Taken all together we get

$$|U||V|\det M_1 > 1 - \varepsilon' \frac{e^{4\delta}}{(\alpha - e^{2\delta})^2} \rightarrow 1, \quad \text{as } b_0 \rightarrow \infty,$$

by Proposition 4.10. In particular,  $\det M_1 > 0$  for  $b_0$  large enough.  $\square$

**Corollary 9.11.** *There exists  $k > 0$  such that if  $f$  is as above, then*

$$\|M_1 x\| \geq k \cdot \min\{|U|^{-1}, |V|^{-1}\} \cdot \|x\|.$$

*Proof.* Write  $M_1$  as

$$M_1 = \begin{pmatrix} \frac{a}{|U|} & -\frac{b}{|V|} \\ -\frac{c}{|U|} & \frac{d}{|V|} \end{pmatrix}.$$

(Here we have used that the distortion of  $\Phi$  and  $\Psi$  are small, so  $D\Phi/D\Psi \asymp |V|/|U|$ .) Then

$$M_1^{-1} = (ad - bc)^{-1} \begin{pmatrix} d|U| & b|U| \\ c|V| & a|V| \end{pmatrix}.$$

We are using the max-norm, hence

$$\|M_1^{-1}\| = (ad - bc)^{-1} \cdot \max\{(b + d)|U|, (c + a)|V|\}.$$

It can be checked that  $(b + d)/(ad - bc)$  and  $(a + c)/(ad - bc)$  are bounded by some  $K$ . Let  $k = 1/K$  to finish the proof.  $\square$

**Proposition 9.12.** *If  $f \in \mathcal{K} \cap \mathcal{L}_\Omega^S$  and  $1 - c_1^+(\mathcal{R}f) \geq \lambda$  for some  $\lambda \in (0, 1)$  (not depending on  $f$ ), then*

$$\begin{aligned} \partial_c u' &\asymp -|C|^{-1}, & \partial_c v' &\asymp |C|^{-1}, & \partial_c c' &\asymp -c'\varepsilon'|C|^{-1}, \\ \partial_u c' &\asymp c'\varepsilon'|U|^{-1}, & \partial_v c' &\asymp -c'\varepsilon'|V|^{-1}. \end{aligned}$$

*Proof.* A straightforward calculation shows that

$$(57) \quad \frac{\partial_c Q_0(x)}{Q'_0(x)} = -\frac{x}{c} \quad \text{and} \quad \frac{\partial_c Q_1(x)}{Q'_1(x)} = -\frac{1-x}{1-c}.$$

This together with  $\Phi \circ Q_0(p) = p$ ,  $\Psi \circ Q_1(q) = q$ , (42) and (46) gives

$$\partial_c p = \frac{\frac{p}{c} Df^{a+1}(p) - \partial_c \Phi(Q_0(p))}{Df^{a+1}(p) - 1}, \quad \partial_c q = \frac{\frac{1-q}{\varepsilon} Df^{b+1}(q) - \partial_c \Psi(Q_1(q))}{Df^{b+1}(q) - 1}.$$

From (44) and (57) we get

$$\begin{aligned} \partial_c \Phi(x) &= -\frac{1}{\varepsilon} \sum_{i=0}^{a-1} Df^{a-i}(x_i) \cdot (1 - x_i), & x_i &= f^i \circ \phi(x), \quad x \in U, \\ \partial_c \Psi(x) &= -\frac{1}{c} \sum_{i=0}^{b-1} Df^{b-i}(x_i) \cdot x_i, & x_i &= f^i \circ \psi(x), \quad x \in V. \end{aligned}$$

Using a similar argument as in the proof of Proposition 9.8 this shows that

$$\partial_c \Phi(x) \asymp -a \quad \text{and} \quad \partial_c \Psi(x) = -\mathcal{O}(b\varepsilon^{1-1/\alpha}),$$

and hence  $\partial_c p \asymp 1$  and  $\partial_c q \asymp 1$ .

Now apply Lemma 9.6 using the fact that  $\Phi^{-1}(p) = Q_0(p)$  to get

$$|U| \partial_c u' = -(1 - u') \partial_c (Q_0(p)) - u' \partial_c (\Phi^{-1}(q)).$$

A calculation gives

$$\partial_c (Q_0(p)) = \frac{Df^{a+1}(p) \left( \frac{p}{c} - \partial_c \Phi(Q_0(p)) \right)}{D\Phi(Q_0(p)) (Df^{a+1}(p) - 1)} \asymp \frac{1}{D\Phi(Q_0(p))}$$

and

$$\partial_c (\Phi^{-1}(q)) = \frac{\partial_c q - \partial_c \Phi(\Phi^{-1}(q))}{D\Phi(\Phi^{-1}(q))} \asymp \frac{1}{D\Phi(\Phi^{-1}(q))}.$$

(In particular, both terms have the same sign.) But  $D\Phi(x) \asymp |C|/|U|$ , so this gives  $\partial_c u' \asymp -|C|^{-1}$ . The proof that  $\partial_c v' \asymp |C|^{-1}$  is almost identical.

From Lemma 9.6 we get

$$|C| \partial_c c' = c'(1 - \partial_c q) + \varepsilon'(1 - \partial_c p),$$

and hence

$$\begin{aligned}\partial_c c' &= \frac{c' \varepsilon' Df^{b+1}(q) - \varepsilon |C|^{-1} (1 - \partial_c \Psi(Q_1(q)))}{\varepsilon \frac{Df^{b+1}(q) - 1}{Df^{b+1}(q) - 1}} \\ &\quad + \frac{\varepsilon' c' Df^{a+1}(p) - c |C|^{-1} (1 - \partial_c \Phi(Q_0(p)))}{c \frac{Df^{a+1}(p) - 1}{Df^{a+1}(p) - 1}} \\ &= -\frac{c' (1 - \partial_c \Psi(Q_1(q)))}{|C| (Df^{b+1}(q) - 1)} - \frac{\varepsilon' (1 - \partial_c \Phi(Q_0(p)))}{|C| (Df^{a+1}(p) - 1)} + \mathcal{O}(c' \varepsilon' / \varepsilon).\end{aligned}$$

From Remark 9.9 we know that  $Df^{a+1}(p) \asymp \alpha u' / c'$  and  $Df^{b+1}(q) \asymp \alpha v' / \varepsilon'$ . Note that  $u' \approx 1$  for  $f \in \mathcal{K} \cap \mathcal{L}_\Omega^S$ , but that  $v'$  can in general be small (this happens if  $f$  renormalizes to a map whose right branch is trivial). However, the assumption that  $1 - c_1^+(\mathcal{R}f) \geq \lambda$  implies that  $v' \geq e^{-\delta} \lambda$  and hence we may assume that  $v' / \varepsilon' \gg 1$  (by increasing  $b_0$  if necessary). Thus,  $Df^{a+1}(p) \asymp \alpha / c'$  and  $Df^{b+1}(q) \asymp \alpha / \varepsilon'$  and by plugging this into the above equation we get  $\partial_c c' \asymp -c' \varepsilon' |C|^{-1}$ . (Note that  $\partial_c \Phi(x) < 0$  and  $\partial_c \Psi(x) < 0$  so there is no cancellation happening.)

Apply Lemma 9.6 to get

$$|C| \partial_u c' = -c' \partial_u q - \varepsilon' \partial_u p.$$

This and the proof of Proposition 9.8 shows that

$$\partial_u c' = \frac{c' (Df^{a+1}(p) - 1) \partial_u \Psi(Q_1(q)) + \varepsilon' (Df^{b+1}(q) - 1) D\Phi(Q_0(p)) \partial_u Q_0(p)}{|C| (Df^{a+1}(p) - 1) (Df^{b+1}(q) - 1)}.$$

Since  $c' (Df^{a+1}(p) - 1) \asymp \alpha - c'$ ,  $\varepsilon' (Df^{b+1}(q) - 1) \asymp \alpha - \varepsilon'$ ,  $|\partial_u \Psi| \ll |D\Phi|$ , and  $\partial_u Q_0(p) \approx 1$ , this shows that

$$\partial_u c' \asymp c' \varepsilon' \frac{D\Phi(Q_0(p))}{|C|} \asymp \frac{c' \varepsilon'}{|U|}.$$

The proof that  $\partial_v c' \asymp -c' \varepsilon' |V|^{-1}$  is almost identical.  $\square$

**Notation.** We need some new notation to state the remaining propositions. Each pure map  $\phi_\sigma$  in the decomposition  $\bar{\phi}$  can be identified with a real number which we denote  $s_\sigma \in \mathbb{R}$ , and each  $\psi_\tau$  in the decomposition  $\bar{\psi}$  can be identified with a real number  $t_\tau \in \mathbb{R}$ :

$$\mathbb{R} \ni s_\sigma \leftrightarrow \phi_\sigma = \bar{\phi}(\sigma) \in \mathcal{Q}, \quad \mathbb{R} \ni t_\tau \leftrightarrow \psi_\tau = \bar{\psi}(\tau) \in \mathcal{Q}.$$

We put primes on these numbers to denote that they come from the renormalization, so  $s'_{\sigma'} \in \mathbb{R}$  is identified with  $\bar{\phi}'(\sigma')$  and  $t'_{\tau'} \in \mathbb{R}$  is identified with  $\bar{\psi}'(\tau')$ . Note that  $\sigma, \sigma'$  are used to denote times for  $\bar{\phi}, \bar{\phi}'$ , and  $\tau, \tau'$  are used to denote times for  $\bar{\psi}, \bar{\psi}'$ , respectively.

**Proposition 9.13.** *There exists  $K$  such that if  $\bar{f} \in \bar{\mathcal{K}} \cap \bar{\mathcal{L}}_\Omega$ , then*

$$\begin{aligned}|\partial_u s'_{\sigma'}| &\leq K \frac{|s'_{\sigma'}|}{|U|}, & |\partial_v s'_{\sigma'}| &\leq K \frac{|s'_{\sigma'}|}{|V|}, & |\partial_c s'_{\sigma'}| &\leq K \frac{|s'_{\sigma'}|}{|C|}, \\ |\partial_u t'_{\tau'}| &\leq K \frac{|t'_{\tau'}|}{|U|}, & |\partial_v t'_{\tau'}| &\leq K \frac{|t'_{\tau'}|}{|V|}, & |\partial_c t'_{\tau'}| &\leq K \frac{|t'_{\tau'}|}{|C|}.\end{aligned}$$

*Proof.* We will compute  $\partial_v s'_{\sigma'}$ ; the other calculations are almost identical. There are four cases to consider depending on which time in the decomposition  $\bar{\phi}'$  we are looking at: (1)  $\bar{\phi}'(\sigma') = Z(\phi_\sigma; I)$ , (2)  $\bar{\phi}'(\sigma') = Z(\psi_\tau; I)$ , (3)  $\bar{\phi}'(\sigma') = Z(Q_0; I)$ , (4)  $\bar{\phi}'(\sigma') = Z(Q_1; I)$ . In each case let  $I = [x, y]$  and let  $T : I \rightarrow C$  be the ‘transfer

map' to  $C$ . This means that  $T = f^i \circ \gamma$  for some  $i$  and  $\gamma$  is a partial composition (e.g.  $\gamma = O_{\geq \sigma}(\bar{\phi})$  in case 1) or a pure map (in cases 3 and 4).

In case 1 Lemma 9.7 gives

$$\partial_v s'_{\sigma'} = \frac{N\phi_{\sigma}(y)}{DT(y)}(\partial_v q - \partial_v T(y)) - \frac{N\phi_{\sigma}(x)}{DT(x)}(\partial_v p - \partial_v T(x)).$$

By Lemma B.14  $N\phi_{\sigma}(y) = N\phi'_{\sigma'}(1)/|I|$  and hence

$$\frac{N\phi_{\sigma}(y)}{DT(y)} \asymp \frac{N\phi'_{\sigma'}(1)/|I|}{|C|/|I|} \asymp \frac{s'_{\sigma'}}{|C|}.$$

Here we have used that the nonlinearity of  $\phi'_{\sigma'}$  does not change sign so  $s'_{\sigma'} = \int N\phi'_{\sigma'}$  and that  $\int N\phi'_{\sigma'} \approx N\phi'_{\sigma'}(1)$  since the nonlinearity is close to being constant (which is true since  $\bar{\phi}'$  is pure and has very small norm).

We now need to estimate  $\partial_v T$  but this can very roughly be bounded by  $\partial_v \Phi$  since

$$\partial_v T(y) = \partial_v f_1^i(\gamma(y)),$$

so the estimate that was used for  $\partial_v \Phi$  in the proof of Proposition 9.8 can be employed. From the same proof we thus get that  $\partial_v q$  dominates both  $\partial_v p$  and  $\partial_v T$ .

The above arguments show that

$$\partial_v s'_{\sigma'} \asymp \frac{s'_{\sigma'}}{|C|} \partial_v q \asymp -\frac{s'_{\sigma'}}{|C|} \frac{D\Psi(Q_1(q))}{Df^{b+1}(q) - 1} \asymp -\frac{s'_{\sigma'}}{|V|} \frac{1}{Df^{b+1}(q) - 1}.$$

This concludes the calculations for case 1.

Case 2 is almost identical to case 1. Case 4 differs in that Lemma 9.7 now gives two extra terms

$$\begin{aligned} \partial_v s'_{\sigma'} &= \frac{NQ_1(y)}{DT(y)}(\partial_v q - \partial_v T(y)) - \frac{NQ_1(x)}{DT(x)}(\partial_v p - \partial_v T(x)) \\ &\quad + \frac{\partial_v Q'_1(y)}{Q'_1(y)} - \frac{\partial_v Q'_1(x)}{Q'_1(x)}. \end{aligned}$$

However,  $\partial_v Q_1 = 1/v$  so the last two terms cancel. The rest of the calculations go exactly like in case 1. Case 3 is similar to case 4.  $\square$

*Remark 9.14.* A key point in the above proof is that deformations in a decomposition direction is monotone. This is what allowed us to estimate the partial derivatives of the 'transfer map'  $T$  by the partial derivatives of  $\Phi$  or  $\Psi$ .

**Proposition 9.15.** *There exists  $K$  and  $\rho > 0$  such that if  $\bar{f} \in \bar{\mathcal{K}} \cap \bar{\mathcal{L}}_{\Omega}$  and  $1 - c_1^+(\mathcal{R}\bar{f}) \geq \lambda$  for some  $\lambda \in (0, 1)$  (not depending on  $\bar{f}$ ), then*

$$\begin{aligned} |\partial_{\star} u'| &\leq \frac{K\varepsilon^{\rho}}{|C|}, & |\partial_{\star} v'| &\leq \frac{K\varepsilon^{\rho}}{|C|}, & |\partial_{\star} c'| &\leq \frac{Kc'\varepsilon'\varepsilon^{\rho}}{|C|}, \\ |\partial_{\star} s'_{\sigma'}| &\leq \frac{K\varepsilon^{\rho}|s'_{\sigma'}|}{|C|}, & |\partial_{\star} t'_{\tau'}| &\leq \frac{K\varepsilon^{\rho}|t'_{\tau'}|}{|C|}, \end{aligned}$$

for  $\star \in \{s_{\sigma}, t_{\tau}\}$ .

*Proof.* Let us first consider  $\partial_{s_{\sigma}}$ , that is deformations in the direction of  $\phi_{\sigma}$ . Since  $\phi_{\sigma}$  is pure we can use (39) to compute

$$(58) \quad \partial_{s_{\sigma}} \phi_{\sigma}(x) \asymp -x(1-x).$$

From (46) we get

$$\partial_{s_\sigma} p = -\frac{\partial_{s_\sigma} \Phi(Q_0(p))}{Df^{a+1}(p) - 1} \quad \text{and} \quad \partial_{s_\sigma} q = -\frac{\partial_{s_\sigma} \Psi(Q_1(q))}{Df^{b+1}(q) - 1}.$$

so the first thing to do is to calculate the partial derivatives of  $\Phi$  and  $\Psi$ .

Let  $x \in U$ , then

$$\begin{aligned} \partial_{s_\sigma} \Phi(x) &= \partial_{s_\sigma} (f_1^a \circ O_{>\sigma}(\bar{\phi}) \circ \phi_\sigma \circ O_{<\sigma}(\bar{\phi}))(x) \\ &= D(f_1^a \circ O_{>\sigma}(\bar{\phi}))(O_{\leq\sigma}(\bar{\phi})(x)) \cdot \partial_{s_\sigma} \phi_\sigma(O_{<\sigma}(\bar{\phi})(x)). \end{aligned}$$

Note that we have used that  $f_1$  does not depend on  $s_\sigma$ . From (58) we thus get that

$$(59) \quad |\partial_{s_\sigma} \Phi(x)| \leq K' \cdot D\Phi(x)(1-x) \leq K\varepsilon.$$

Let  $x \in V$  and let  $x_i = f_0^i \circ \psi(x)$ . As in the proof of Proposition 9.8 we have

$$\partial_{s_\sigma} \Psi(x) = \partial_{s_\sigma} f_0(x_{b-1}) + \sum_{i=1}^{b-1} Df_0^{b-i}(x_i) \partial_{s_\sigma} f_0(x_{i-1}).$$

From (58) we get

$$\begin{aligned} |\partial_{s_\sigma} f_0(x_{i-1})| &= |D(O_{>\sigma}(\bar{\phi}))(O_{\leq\sigma}(\bar{\phi}) \circ Q_0(x_{i-1})) \cdot \partial_{s_\sigma} (O_{<\sigma}(\bar{\phi}) \circ Q_0(x_{i-1}))| \\ &\leq K|x_i|. \end{aligned}$$

Using the same estimate as in the proof of Proposition 9.8 this shows that

$$(60) \quad |\partial_{s_\sigma} \Psi(x)| \leq K'(1-x_b) + \mathcal{O}(b\varepsilon^{1-1/\alpha}) = \mathcal{O}(b\varepsilon^{1-1/\alpha}).$$

We can now argue as in the proof of Proposition 9.8 to find bounds on  $\partial_{s_\sigma} \star$  for  $\star \in \{u', v', c'\}$ . From Lemma 9.6 we get

$$\begin{aligned} \partial_{s_\sigma} u' &= \frac{1-u'}{|U|} \cdot \frac{\partial_{s_\sigma} \Phi(Q(p))}{D\Phi(Q(p))} \cdot \frac{Df^{a+1}(p)}{Df^{a+1}(p) - 1} + \frac{u'}{|U|} \frac{\partial_{s_\sigma} \Phi(\Phi^{-1}(q)) - \partial_{s_\sigma} q}{D\Phi(\Phi^{-1}(q))}, \\ -\partial_{s_\sigma} v' &= \frac{1-v'}{|V|} \cdot \frac{\partial_{s_\sigma} \Psi(Q(q))}{D\Psi(Q(q))} \cdot \frac{Df^{b+1}(q)}{Df^{b+1}(q) - 1} + \frac{v'}{|V|} \frac{\partial_{s_\sigma} \Psi(\Psi^{-1}(p)) - \partial_{s_\sigma} p}{D\Psi(\Psi^{-1}(p))}, \\ \partial_{s_\sigma} c' &= c' \cdot \frac{\partial_{s_\sigma} \Psi(Q_1(q))}{Df^{b+1}(q) - 1} + \varepsilon' \cdot \frac{\partial_{s_\sigma} \Phi(Q_0(p))}{Df^{a+1}(p) - 1}. \end{aligned}$$

Use that  $D\phi \asymp |C|/|U|$ ,  $D\Psi \asymp |C|/|V|$ ,  $Df^{a+1}(p) \asymp \alpha/c'$  and  $Df^{b+1}(q) \asymp \alpha/\varepsilon'$  (see the proof of Proposition 9.12) to finish the estimates for  $\partial_{s_\sigma} u'$ ,  $\partial_{s_\sigma} v'$  and  $\partial_{s_\sigma} c'$ . Note that  $b\varepsilon^r \rightarrow 0$  for any  $r > 0$  so it is clear from (59) and (60) that we can find a  $\rho > 0$  such that  $|\partial_{s_\sigma} \Phi| < K\varepsilon^\rho$  and  $|\partial_{s_\sigma} \Psi| < K\varepsilon^\rho$ .

In order to find bounds for  $\star \in \{s'_{\sigma'}, t'_{\tau'}\}$  we argue as in the proof of Proposition 9.13. The last two terms from Lemma 9.7 are slightly different (when nonzero). In this case they are given by

$$\frac{\partial_{s_\sigma} (D\phi_\sigma)(y)}{D\phi_\sigma(y)} - \frac{\partial_{s_\sigma} (D\phi_\sigma)(x)}{D\phi_\sigma(x)}.$$

Using (58) we can calculate this difference. For  $|s_\sigma| \ll 1$  it is close to  $y - x$  which turns out to be negligible. All other details are exactly like the proof of Proposition 9.13.

The estimates for  $\partial_{t_\tau}$  are handled similarly. The only difference is the estimates of the partial derivatives of  $\Phi$  and  $\Psi$ . These can be determined by arguing as in the above and the proof of Proposition 9.8 which results in

$$(61) \quad |\partial_{t_\tau} \Phi(x)| \leq K\varepsilon^{1-1/\alpha} \quad \text{and} \quad |\partial_{t_\tau} \Psi(y)| \leq Ka\varepsilon,$$

for  $x \in U$  and  $y \in V$ . The remaining estimates are handled identically to the above.  $\square$

## 10. INVARIANT CONE FIELD

A standard way of showing hyperbolicity of a linear map is to find an invariant cone field with expansion inside the cones and contraction in the complement of the cones. In this section we show that the derivative of the renormalization operator has an invariant cone field and that it expands these cones. However, our estimates on the derivative are not sufficient to prove contraction in the complement of the cones so we cannot conclude that the derivative is hyperbolic. The results in this section are used in Section 11 to study the structure of the parameter plane and in Section 12 to construct unstable manifolds in the limit set of renormalization.

Let

$$H(\bar{f}, \kappa) = \{(x, y) \mid \|y\| \leq \kappa\|x\|\}$$

denote the standard horizontal  $\kappa$ -cone on the tangent space at  $\bar{f}$ . Recall that we decompose the tangent space into a two-dimensional subspace with coordinate  $x$  and a codimension two subspace with coordinate  $y$ . The  $x$ -coordinate corresponds to the  $(u, v)$ -subspace in  $\bar{\mathcal{L}}$ . We use the max-norm so if  $z = (x, y)$  then  $\|z\| = \max\{\|x\|, \|y\|\}$ .

**Proposition 10.1.** *Assume  $\bar{f} \in \bar{\mathcal{K}} \cap \bar{\mathcal{L}}_\Omega$  and  $1 - c_1^+(\mathcal{R}\bar{f}) \geq \lambda$  for some  $\lambda \in (0, 1)$  (not depending on  $\bar{f}$ ). Define*

$$\kappa^-(\bar{f}) = K^- \max\{\varepsilon, \text{Dist } \bar{\phi}, \text{Dist } \bar{\psi}\} \quad \text{and} \quad \kappa^+(\bar{f}) = K^+ \min\left\{\frac{|C|}{|U|}, \frac{|C|}{|V|}\right\}.$$

*It is possible to choose  $K^+$ ,  $K^-$  (not depending on  $\bar{f}$ ) such that if  $\kappa \leq \kappa^+(\bar{f})$ , then*

$$D\mathcal{R}_{\bar{f}}(H(\bar{f}, \kappa)) \subset H(\mathcal{R}\bar{f}, \kappa^-(\mathcal{R}\bar{f})),$$

*for  $b_0$  large enough. In particular, the cone field  $\bar{f} \mapsto H(\bar{f}, 1)$  is mapped strictly into itself by  $D\mathcal{R}$ .*

*Remark 10.2.* Note that as  $b_0$  increases,  $\kappa^- \downarrow 0$  and  $\kappa^+ \uparrow \infty$ . Thus a fatter and fatter cone is mapped into a thinner and thinner cone. In particular, the invariant subspaces inside the thin cone and the complement of the fat cone eventually line up with the coordinate axes.

*Proof.* Assume  $\|y\| \leq \kappa\|x\|$ . Let  $z' = Mz$ , where  $M = D\mathcal{R}_{\bar{f}}$  as in (41),  $z' = (x', y')$  and  $z = (x, y)$ . Then

$$\frac{\|x'\|}{\|y'\|} \geq \frac{\|M_1x\| - \|M_2\|\|y\|}{\|M_3x\| + \|M_4\|\|y\|} \geq \frac{\|M_1\frac{x}{\|x\|}\| - \kappa\|M_2\|}{\|M_3\frac{x}{\|x\|}\| + \kappa\|M_4\|}.$$

We are interested in a lower bound on  $\|x'\|/\|y'\|$  so this shows that we need to minimize

$$g(x) = \frac{\|M_1x\| - \kappa\|M_2\|}{\|M_3x\| + \kappa\|M_4\|},$$



subject to the constraint  $\|x\| = \max\{|x_1|, |x_2|\} = 1$ . We can write  $M_1$  on the form

$$M_1 = \begin{pmatrix} \frac{m_{11}}{|U|} & -\frac{m_{12}}{|V|} \\ -\frac{m_{21}}{|U|} & \frac{m_{22}}{|V|} \end{pmatrix},$$

where the entries  $m_{ij}$  are positive, bounded, and  $m_{ii} \geq 1$ , by Proposition 9.8. Furthermore, by Theorem 9.2 we know that

$$\|M_3 x\| \leq K\rho' \left( \frac{|x_1|}{|U|} + \frac{|x_2|}{|V|} \right).$$

Hence, if  $|x_1| = 1$ , then

$$g(x) \geq \frac{\max \left\{ \left| \frac{m_{11}}{|U|} - \frac{m_{12}x_2}{|V|} \right|, \left| \frac{m_{21}}{|U|} - \frac{m_{22}x_2}{|V|} \right| \right\} - \kappa\|M_2\|}{K\rho' \left( \frac{1}{|U|} + \frac{|x_2|}{|V|} \right) + \kappa\|M_4\|} = g_1(x_2)$$

and if  $|x_2| = 1$ , then

$$g(x) \geq \frac{\max \left\{ \left| \frac{m_{11}x_1}{|U|} - \frac{m_{12}}{|V|} \right|, \left| \frac{m_{21}x_1}{|U|} - \frac{m_{22}}{|V|} \right| \right\} - \kappa\|M_2\|}{K\rho' \left( \frac{|x_1|}{|U|} + \frac{1}{|V|} \right) + \kappa\|M_4\|} = g_2(x_1)$$

Thus we are interested in minimizing  $g_i(t)$  for  $i = 1, 2$  and  $t \in [0, 1]$  (note that  $g_i(-t) \geq g_i(t)$  for  $t \in [0, 1]$  so we do not need to consider negative  $t$ ).

The maps  $g_i$  are piecewise Möbius maps (which are also nonsingular); in particular, they are piecewise monotone so any minimum is assumed at 0, 1, or at a boundary of monotonicity. A boundary of monotonicity can only occur when the two terms inside the max term in the numerator are equal. By solving the equations

$$\frac{m_{11}}{|U|} - \frac{m_{12}t}{|V|} = \pm \left( \frac{m_{21}}{|U|} - \frac{m_{22}t}{|V|} \right)$$

we see that  $g_1$  has (at most) two points,  $t_-$  and  $t_+$ , where it is not monotone on any neighborhood. These points are

$$t_- = \frac{|V|}{|U|} \frac{m_{11} - m_{21}}{m_{12} - m_{22}} \quad \text{and} \quad t_+ = \frac{|V|}{|U|} \frac{m_{11} + m_{21}}{m_{12} + m_{22}}.$$

From similar considerations we see that  $g_2$  has (at most) two points where it is not monotone on any neighborhood, namely  $t_-^{-1}$  and  $t_+^{-1}$ . Note that we say “at most” here since we do not know if  $t_{\pm} \in [0, 1]$  or if  $t_{\pm}^{-1} \in [0, 1]$ , nor will it turn out to matter.

Thus, to minimize  $g(x)$  we only have to find the minimum of  $g_1(0)$ ,  $g_2(0)$ ,  $g_1(1) = g_2(1)$ ,  $g_1(t_{\pm})$  and  $g_2(t_{\pm}^{-1})$ . We will calculate these values one at a time.

Consider  $g_1(0)$  first. From Theorem 9.2 we get that<sup>11</sup>

$$\|M_2\| \leq K_1/|C| \quad \text{and} \quad \|M_4\| \leq K_2\rho'/|C|,$$

and hence

$$g_1(0) = \frac{\max\{m_{11}, m_{21}\} - \kappa\|M_2\||U|}{K\rho' + \kappa\|M_4\||U|} \geq \frac{1 - \kappa K_1|U|/|C|}{K\rho' + \kappa K_2\rho'|U|/|C|}.$$

<sup>11</sup>This is the only place where the condition on  $c_1^+(\mathcal{R}f)$  is used. It is necessary to get the  $\rho'$  term in the bound on  $\|M_4\|$ .

In the inequality we used the fact that  $m_{11} \geq 1$ . Hence

$$(62) \quad \kappa \leq \frac{|C|}{2K_1|U|} \implies g_1(0) \geq \frac{1}{\rho'(2K + K_2/K_1)}.$$

Consider  $g_2(0)$ :

$$g_2(0) = \frac{\max\{m_{12}, m_{22}\} - \kappa\|M_2\||V|}{K\rho' + \kappa\|M_4\||V|} \geq \frac{1 - \kappa K_1|V|/|C|}{K\rho' + \kappa K_2\rho'|V|/|C|}.$$

In the inequality we used Theorem 9.2 and the fact that  $m_{22} \geq 1$ . Hence

$$(63) \quad \kappa \leq \frac{|C|}{2K_1|V|} \implies g_2(0) \geq \frac{1}{\rho'(2K + K_2/K_1)}.$$

Consider  $g_1(t_\pm)$ :

$$\begin{aligned} g_1(t_\pm) &= \frac{|m_{11} - m_{12} \frac{m_{11} \pm m_{21}}{m_{12} \pm m_{22}}| - \kappa\|M_2\||U|}{K\rho' \left(1 + \left|\frac{m_{11} \pm m_{21}}{m_{12} \pm m_{22}}\right|\right) + \kappa\|M_4\||U|} \\ &= \frac{|m_{11}m_{22} - m_{12}m_{21}| - \kappa\|M_2\||U||m_{12} \pm m_{22}|}{K\rho' (|m_{11} \pm m_{21}| + |m_{12} \pm m_{22}|) + \kappa\|M_4\||U||m_{11} \pm m_{21}|}. \end{aligned}$$

There exists  $\nu$  such that  $\sum m_{ij} \leq \nu$  and by Corollary 9.10 there exists  $\mu > 0$  such that  $m_{11}m_{22} - m_{12}m_{21} \geq \mu$ , so

$$g_1(t_\pm) \geq \frac{\mu - \kappa\nu K_1|U|/|C|}{K\rho'\nu + \kappa\nu K_2\rho'|U|/|C|},$$

where we once again have used Theorem 9.2. Hence

$$(64) \quad \kappa \leq \frac{\mu|C|}{2K_1\nu|U|} \implies g_1(t_\pm) \geq \frac{1}{\rho' \left( \frac{2K\nu}{\mu} + \frac{K_2}{K_1} \right)}.$$

An almost identical calculation for  $g_2(t_\pm^{-1})$  results in:

$$(65) \quad \kappa \leq \frac{\mu|C|}{2K_1\nu|V|} \implies g_2(t_\pm^{-1}) \geq \frac{1}{\rho' \left( \frac{2K\nu}{\mu} + \frac{K_2}{K_1} \right)}.$$

Finally, consider  $g_1(1)$ :

$$g_1(1) = \frac{\max \left\{ \left| \frac{m_{11}}{|U|} - \frac{m_{12}}{|V|} \right|, \left| \frac{m_{21}}{|U|} - \frac{m_{22}}{|V|} \right| \right\} - \kappa\|M_2\|}{K\rho' \left( \frac{1}{|U|} + \frac{1}{|V|} \right) + \kappa\|M_4\|}.$$

We need to minimize the numerator, so introduce a variable  $s$  and assume that

$$\frac{m_{11}}{|U|} = s \frac{m_{12}}{|V|}, \quad s \in \mathbb{R}.$$

Let  $H(s) = \max\{h_1(s), h_2(s)\}$ , where

$$\begin{aligned} h_1(s) &= \left| \frac{m_{11}}{|U|} - \frac{m_{12}}{|V|} \right| = \frac{m_{12}}{|V|} |s - 1|, \\ h_2(s) &= \left| \frac{m_{21}}{|U|} - \frac{m_{22}}{|V|} \right| = \frac{1}{|V|} \left| \frac{m_{12}m_{21}}{m_{11}} s - m_{22} \right|. \end{aligned}$$

The equation  $H(s) = 0$  has two solutions:  $s_1 = 1$  and  $s_2 = m_{11}m_{22}/(m_{12}m_{21})$ . Note that  $h_i$  is decreasing to the left of  $s_i$  and increasing to the right of  $s_i$ , for

$i = 1, 2$ . Also,  $s_1 < s_2$  by Corollary 9.10 so  $H(s)$  assumes its minimum at  $s_*$ , where  $s_*$  is defined by  $h_1(s_*) = h_2(s_*)$  and  $s_1 < s_* < s_2$ . Solving this equation gives

$$s_* = \frac{m_{11}(m_{12} + m_{22})}{m_{12}(m_{11} + m_{21})}.$$

Thus

$$\min H(s) = H(s_*) = \frac{m_{11}m_{22} - m_{12}m_{21}}{|V|(m_{11} + m_{21})}.$$

Note that we can also write  $h_1(s) = m_{11}|1 - s^{-1}|/|U|$  and thus

$$\min H(s) = h_1(s_*) = \frac{m_{11}m_{22} - m_{12}m_{21}}{|U|(m_{12} + m_{22})},$$

which shows that

$$H(s) \geq \frac{\mu}{\nu} \max\{|U|^{-1}, |V|^{-1}\}.$$

Putting all of this together, we arrive at

$$g_1(1) \geq \frac{\frac{\mu}{\nu} \max\{|U|^{-1}, |V|^{-1}\} - \kappa K_1/|C|}{K\rho'2 \max\{|U|^{-1}, |V|^{-1}\} + \kappa\rho'K_2/|C|}.$$

Hence

$$(66) \quad \kappa \leq \frac{\mu|C| \max\{|U|^{-1}, |V|^{-1}\}}{2\nu K_1} \implies g_1(1) \geq \frac{1}{\rho' \left( \frac{4\nu K}{\mu} + \frac{K_2}{K_1} \right)}.$$

From (62), (63), (64), (65) and (66) we get that, if  $\|y\| \leq \kappa\|x\|$  and

$$\kappa \leq \frac{\min\{1, \mu/\nu\}}{2K_1} \min \left\{ \frac{|C|}{|U|}, \frac{|C|}{|V|} \right\},$$

then

$$\|y'\| \leq \|x'\| \rho'(2K \max\{1, 2\nu/\mu\} + K_2/K_1). \quad \square$$

**Proposition 10.3.** *Let  $\bar{f} \in \bar{\mathcal{K}} \cap \bar{\mathcal{L}}_\Omega$  and  $1 - c_1^+(\mathcal{R}\bar{f}) \geq \lambda$  for some  $\lambda \in (0, 1)$  (not depending on  $\bar{f}$ ). Then  $D\mathcal{R}$  is strongly expanding on the cone field  $\bar{f} \mapsto H(\bar{f}, 1)$ . Specifically, there exists  $k > 0$  (not depending on  $\bar{f}$ ) such that*

$$\|D\mathcal{R}_{\bar{f}}z\| \geq k \cdot \min\{|U|^{-1}, |V|^{-1}\} \cdot \|z\|, \quad \forall z \in H(\bar{f}, 1) \setminus \{0\}.$$

*Proof.* Use Corollary 9.11 to get

$$\|Mz\| \geq \|M_1x + M_2y\| \geq |k \cdot \min\{|U|^{-1}, |V|^{-1}\} - \|M_2\|| \cdot \|x\|.$$

Now use the fact that  $\|z\| = \|x\|$  for  $z \in H(\bar{f}, 1)$  to finish the proof.  $\square$

## 11. ARCHIPELAGOS IN THE PARAMETER PLANE

The term *archipelago* was introduced by Martens and de Melo [2001] to describe the structure of the domains of renormalizability in the parameter plane for families of Lorenz maps. In this section we show how the information we have on the derivative of the renormalization operator can be used to prove that the structure of archipelagos must be very rigid.

Fix  $c_*$ ,  $\phi_*$ ,  $\psi_*$  and let  $F : [0, 1]^2 \rightarrow \mathcal{L}$  denote the associated family of Lorenz maps

$$(u, v) = \lambda \mapsto F_\lambda = (u, v, c_*, \phi_*, \psi_*).$$

We will assume that  $F_\lambda \in \mathcal{K} \cap \mathcal{L}_\Omega^S$  (see Definition 4.1) and that  $b_0$  has been fixed (and is large enough).

**Definition 11.1.** An archipelago  $A_\omega \subset [0, 1]^2$  of type  $\omega \in \Omega$  is the set of  $\lambda$  such that  $F_\lambda$  is  $\omega$ -renormalizable. An island of  $A_\omega$  is a connected component of the interior of  $A_\omega$ .

For the family  $\lambda \mapsto F_\lambda$  we have the following very strong structure theorem for archipelagos [this should be contrasted with Martens and de Melo, 2001]. Note that  $c_*$ ,  $\phi_*$  and  $\psi_*$  are arbitrary, so the results in this section holds for *any* family such that  $F_\lambda \in \mathcal{K} \cap \mathcal{L}_\Omega^S$ .

**Theorem 11.2.** *For every  $\omega \in \Omega$  there exists a unique island  $I$  such that the archipelago  $A_\omega$  equals the closure of  $I$ . Furthermore,  $I$  is diffeomorphic to a square.*

*Remark 11.3.* This theorem shows that the structure of  $A_\omega$  is very rigid. Note that the structure of archipelagos is much more complicated in general. There may be multiple islands, islands need not be square, there may be isolated points, etc.

**Theorem 11.4.** *For every  $\bar{\omega} \in \Omega^\mathbb{N}$  there exists a unique  $\lambda$  such that  $F_\lambda$  has combinatorial type  $\bar{\omega}$ . The set of all such  $\lambda$  is a Cantor set.*

The family  $F_\lambda$  is monotone, by which we mean that  $u \mapsto F_{(u,v)}(x)$  is strictly increasing for  $x \in (0, c_*)$ , and  $v \mapsto F_{(u,v)}(x)$  is strictly decreasing for  $x \in (c_*, 1)$ . As a consequence, if we let

$$M_{(u,v)}^+ = \{(x, y) \mid x \geq u, y \leq v\} \quad \text{and} \quad M_{(u,v)}^- = \{(x, y) \mid x \leq u, y \geq v\},$$

then

$$\mu \in M_\lambda^+ \implies F_\mu(x) > F_\lambda(x) \quad \text{and} \quad \mu \in M_\lambda^- \implies F_\mu(x) < F_\lambda(x),$$

for all  $x \in (0, 1) \setminus \{c\}$ . In other words, deformations in  $M_\lambda^+$  moves both branches up, deformations in  $M_\lambda^-$  moves both branches down. This simple observation is key to analyzing the structure of archipelagos.

**Definition 11.5.** Let  $\pi_S : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the projection which takes the rectangle  $[c, 1] \times [1 - c, 1] \times \{c\}$  onto  $S = [1/2, 1]^2$

$$\pi_S(x, y, c) = \left( 1 - \frac{1 - x}{2(1 - c)}, 1 - \frac{1 - y}{2c} \right),$$

and let  $H$  be the map which takes  $(u, v, c, \phi, \psi)$  to the height of its branches ( $c$  is kept around because  $\pi_S$  needs it)

$$H(u, v, c, \phi, \psi) = (\phi(u), 1 - \psi(1 - v), c).$$

Now define  $R : A_\omega \rightarrow S$  by

$$R(\lambda) = \pi_S \circ H \circ \mathcal{R}(F_\lambda).$$

*Remark 11.6.* The action of  $R$  can be understood by looking at Figure 8. The boundary of an island  $I$  is mapped into the boundary of the wedge  $W$  by the map  $H \circ \mathcal{R}$ . The four boundary pieces of the wedge correspond to when the renormalization has at least one full or trivial branch. Note that the image of  $\partial I$  in  $\partial W$  will not in general lie in a plane, instead it will be bent around somewhat. For this reason we project down to the square  $S$  via the projection  $\pi_S$ . This gives us the final operator  $R : A_\omega \rightarrow S$ .

**Proposition 11.7.** *Let  $I \subset A_\omega$  be an island. Then  $R$  is an orientation-preserving diffeomorphism that takes the closure of  $I$  onto  $S$ .*

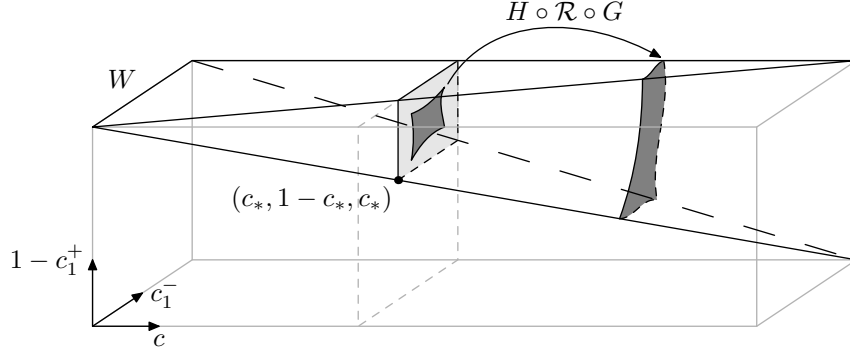


FIGURE 8. Illustration of the action of  $\mathcal{R}$  on the family  $F_\lambda$ . The dark gray island is mapped onto a set which is wrapped around the wedge  $W$ . That is, the boundary of the island is mapped into the boundary of  $W$  with nonzero degree. Note that in this illustration we project the image of  $\mathcal{R}$  to  $\mathbb{R}^3$  via the map  $H$ . The maps  $H$  and  $G$  convert between critical values  $(c_1^-, c_1^+)$  and  $(u, v)$ -parameters. Explicitly  $G(c_1^-, 1 - c_1^+, c_*) = (\phi_*^{-1}(c_1^-), 1 - \psi_*^{-1}(c_1^+), c_*, \phi_*, \psi_*)$ .

*Remark 11.8.* This already shows that the structure of archipelagos is very rigid. First of all every island is full, but there are also exactly one of each type of extremal points, and exactly one of each type of vertex. In other words, there are no degenerate islands of any type! Extremal points and vertices are defined in Martens and de Melo [2001], see also the caption of Figure 9.

*Proof.* By definition  $R$  maps  $I$  into  $S$  and  $\partial I$  into  $\partial S$ . We claim that  $DR_\lambda$  is orientation-preserving for every  $\lambda \in \text{cl } I$ .<sup>12</sup> Assume that the claim holds (we will prove this soon).

We contend that  $R$  maps  $\text{cl } I$  onto  $S$ . If not, then  $R(\partial I)$  must be strictly contained in  $\partial S$ , since the boundaries are homeomorphic to the circle and  $R$  is continuous. But then  $DR_\lambda$  must be singular for some  $\lambda \in \partial I$  which contradicts the claim.

Hence  $R : \text{cl } I \rightarrow S$  maps a simply connected domain onto a simply connected domain, and  $DR$  is a local isomorphism. Thus  $R$  is in fact a diffeomorphism.

We now prove the claim. A computation gives

$$D\pi_S(x, y, c) = \begin{pmatrix} (2(1-c))^{-1} & 0 & \star \\ 0 & (2c)^{-1} & \star \end{pmatrix},$$

and

$$DH_{(u,v,c,\phi,\psi)} = \begin{pmatrix} \phi'(u) & 0 & \dots \\ 0 & \psi'(1-v) & \dots \\ \star & \star & \dots \end{pmatrix}.$$

The top-left  $2 \times 2$  matrix is orientation-preserving in both cases and the same is true for  $DR$  by Corollary 9.10. Thus  $DR_\lambda$  is orientation-preserving.  $\square$

<sup>12</sup>The notation  $DR_\lambda$  is used to denote the derivative of  $R$  at the point  $\lambda$ .

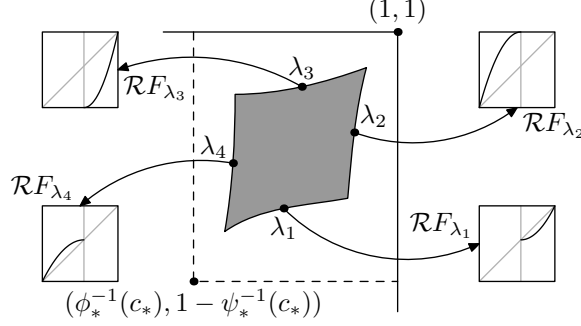


FIGURE 9. Illustration of a full island for the family  $F_\lambda$ . The boundary corresponds to when at least one branch of the renormalization  $\mathcal{R}F_\lambda$  is either full or trivial. The top right and bottom left corners are extremal points; the top left and bottom right corners are vertices.

**Lemma 11.9.** *Assume  $f^m(c_1^-) = c = f^n(c_1^+)$  for some  $m, n > 0$ . Let  $(l, c)$  and  $(c, r)$  be branches of  $f^m$  and  $f^n$ , respectively. Then  $f^m(l) \leq l$  and  $f^n(r) \geq r$ . In particular,  $f$  is renormalizable to a map with trivial branches.*

*Proof.* In order to reach a contradiction we assume that  $f^m(l) > l$ . Then  $f^{im}(l) \uparrow x$  for some point  $x \in (l, c]$  as  $i \rightarrow \infty$ , since  $f^m(c_1^-) = c$ . Since  $l$  is the left endpoint of a branch there exists  $t$  such that  $f^t(l) = c_1^+$ . Hence  $f^{m-t}(c_1^+) = l$  so the orbit of  $c_1^+$  contains the orbit of  $l$ . But the orbit of  $c_1^+$  was periodic by assumption which contradicts  $f^{im}(l) \uparrow x$ . Hence  $f^m(l) \leq l$ .

Now repeat this argument for  $r$  to complete the proof.  $\square$

**Definition 11.10.** Define

$$\begin{aligned} \gamma_{\text{triv}}^- &= \{\lambda \in [0, 1]^2 \mid F_\lambda^{a+1}(c_*^-) = c_* \text{ and } F_\lambda^i(c_*^-) > c_*, i = 1, \dots, a\}, \\ \gamma_{\text{triv}}^+ &= \{\lambda \in [0, 1]^2 \mid F_\lambda^{b+1}(c_*^+) = c_* \text{ and } F_\lambda^i(c_*^+) < c_*, i = 1, \dots, b\}. \end{aligned}$$

(The notation here is  $g(c_*^-) = \lim_{x \uparrow c_*} g(x)$  and  $g(c_*^+) = \lim_{x \downarrow c_*} g(x)$ .)

**Lemma 11.11.** *The set  $\gamma_{\text{triv}}^-$  is the image of a curve  $v \mapsto (g(v), v)$ . The map  $g$  is differentiable and takes  $[1 - \psi_*^{-1}(c_*), 1]$  into  $[\phi_*^{-1}(c_*), 1]$ .*

*Similarly,  $\gamma_{\text{triv}}^+$  is the image of a curve  $u \mapsto (u, h(u))$  where  $h$  is differentiable and takes  $[\phi_*^{-1}(c_*), 1]$  into  $[1 - \psi_*^{-1}(c_*), 1]$ .*

*Proof.* Define

$$g(v) = \phi_*^{-1} \circ (\psi_* \circ Q_1)^{-a}(c_*) \quad \text{and} \quad h(u) = 1 - \psi_*^{-1} \circ (\phi_* \circ Q_0)^{-b}(c_*).$$

Note that  $Q_1$  depends on  $v$  and  $Q_0$  depends on  $u$  so  $g$  and  $h$  are well-defined maps. It can now be checked that these maps define  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$ .  $\square$

**Lemma 11.12.** *Assume that  $\gamma_{\text{triv}}^-$  crosses  $\gamma_{\text{triv}}^+$  and let  $\lambda \in \gamma_{\text{triv}}^- \cap \gamma_{\text{triv}}^+$ . Then the crossing is transversal and there exists  $\rho > 0$  such that if  $r < \rho$ , then the complement of  $\gamma_{\text{triv}}^- \cup \gamma_{\text{triv}}^+$  inside the ball  $B_r(\lambda)$  consists of four components and exactly one of these components is contained in the archipelago  $A_\omega$ .*

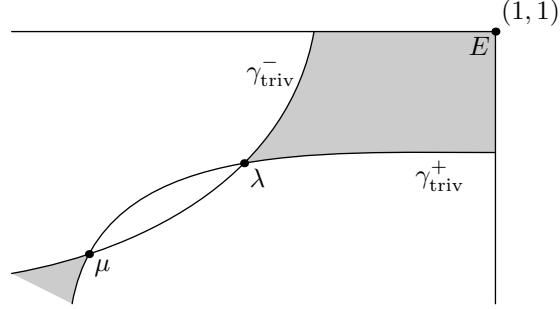


FIGURE 10. Illustration of the proof of Theorem 11.2. Both  $\lambda$  and  $\mu$  must be in the boundary of islands, which lie inside the shaded areas. These two islands have opposite orientation which is impossible.

*Proof.* To begin with assume that the crossing is transversal so that the complement of  $\gamma_{\text{triv}}^- \cup \gamma_{\text{triv}}^+$  in  $B_r(\lambda)$  automatically consists of four components for  $r$  small enough. Note that  $\gamma_{\text{triv}}^- \cup \gamma_{\text{triv}}^+$  does not intersect  $M_\lambda^+ \cup M_\lambda^- \setminus \{\lambda\}$ . Hence, precisely one component will have a boundary point  $\mu \in \gamma_{\text{triv}}^-$  such that  $\gamma_{\text{triv}}^+$  intersects  $M_\mu^+$ . Denote this component by  $N$ . Note that if we move from  $\mu$  inside  $N \cap M_\mu^+$  then the left critical value of the return map moves above the diagonal. If we move in  $N \cap M_\mu^+$  from a point in  $\gamma_{\text{triv}}^+$  then the right critical value of the return map moves below the diagonal.

By Lemma 11.9  $F_\lambda$  is renormalizable and moreover the periodic points  $p_\lambda$  and  $q_\lambda$  that define the return interval of  $F_\lambda$  are hyperbolic repelling by the minimum principle. Hence, if we deform  $F_\lambda$  into  $N$  it will still be renormalizable since  $N$  consists of  $\mu$  such that  $F_\mu^{a+1}(c^-)$  is above the diagonal and  $F_\mu^{b+1}(c^+)$  is below the diagonal. By choosing  $r$  small enough all of  $N$  will be contained in  $A_\omega$ .

Note that if we deform into any other component (other than  $N$ ) then at least one of the critical values of the return map will be on the wrong side of the diagonal and hence the corresponding map is not renormalizable. Thus only the component  $N$  intersects  $A_\omega$ .

Now assume that the crossing is not transversal. Then we may pick  $\lambda$  in the intersection  $\gamma_{\text{triv}}^- \cap \gamma_{\text{triv}}^+$  so that it is on the boundary of an island (by the above argument). But then  $\lambda$  must be at a transversal intersection since islands are square by Proposition 11.7 and the curves  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  are differentiable. Hence every crossing is transversal.  $\square$

*Proof of Theorem 11.2.* From Proposition 11.7 we know that every island must contain an extremal point which renormalizes to a map with only trivial branches, and hence every island must be adjacent to a crossing between the curves  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$ . We claim that there can be only one such crossing and hence uniqueness of islands follows. Note that there is always at least one island by Proposition 6.6.

By Lemma 11.11  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  terminate in the upper and right boundary of  $[0, 1]^2$ , respectively. Let  $\lambda$  be the crossing nearest the points of termination in these boundaries. Let  $E$  be the component in the complement of  $\gamma_{\text{triv}}^- \cup \gamma_{\text{triv}}^+$  in  $[0, 1]^2$  that contains the point  $(1, 1)$ . The geometrical configuration of  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  is

such that  $E$  must contain the piece of  $A_\omega$  adjacent to  $\lambda$  as in Lemma 11.12. To see this use the fact that deformations in the cones  $M_\lambda^+$  moves both branches of  $F_\lambda$  up.

In order to reach a contradiction assume that there exists another crossing  $\mu$  between  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  (see Figure 10). By Lemma 11.12 there is an island attached to this crossing but the configuration of  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  at  $\mu$  is such that this island is oriented opposite to the island inside  $E$ . But  $R$  is orientation-preserving so both islands must be oriented the same way and hence we reach a contradiction. The conclusion is that there can be no more than one crossing between  $\gamma_{\text{triv}}^-$  and  $\gamma_{\text{triv}}^+$  as claimed.

Finally, the entire archipelago equals the closure of the island since the derivative of  $R$  is nonsingular at every point in the archipelago. Hence every point in the archipelago must either be contained in an island or on the boundary of an island.  $\square$

*Proof of Theorem 11.4.* By Theorem 11.2 there exists a unique sequence of nested squares<sup>13</sup>

$$I_1 \supset I_2 \supset I_3 \supset \dots$$

such that  $\lambda \in I_k$  implies that  $F_\lambda$  is renormalizable of type  $(\omega_0, \dots, \omega_{k-1})$ .

Let  $F : [0, 1]^2 \rightarrow \mathcal{L}$  be the map  $\lambda \mapsto F_\lambda$ , let  $p : \mathcal{L} \rightarrow [0, 1]^2$  be the projection  $(u, v, c, \phi, \psi) \mapsto (u, v)$ , and let  $G_k : I_k \rightarrow [0, 1]^2$  be defined by  $G_k = p \circ \mathcal{R}^k \circ F$ . The set of tangent vectors  $v$  to  $F(I_k)$  are all horizontal, so the image of  $v$  under  $D\mathcal{R}$  is in a horizontal cone with a very small angle by Proposition 10.1. This cone is invariant under  $D\mathcal{R}$  by the same proposition and furthermore it is strongly expanded by Proposition 10.3.<sup>14</sup> Proposition 4.5 shows that  $\mathcal{K}$  is relatively compact so there are uniform bounds on  $|U|$  and  $|V|$  and hence Proposition 10.3 shows that there exists  $\mu > 1$  such that

$$\|DG_k\| \geq \mu^k.$$

By construction  $G_k(I_k) \subset [0, 1]^2$  so the image is bounded, which together with the lower bound on  $\|DG_k\|$  shows that the diameter of  $I_k$  shrinks at an exponential rate. In particular  $\bigcap I_k$  is a point. That the union of all such points is a Cantor set is a standard argument (using that each type  $\bar{\omega} \in \Omega^\mathbb{N}$  has a unique associated sequence of squares  $\{I_k\}$  and that each such sequence shrinks at a uniform exponential rate).  $\square$

## 12. UNSTABLE MANIFOLDS

The norm used on the tangent space does not give good enough estimates to see a contracting subspace so we cannot quite prove that the limit set of  $\mathcal{R}$  is hyperbolic. However, these estimates did give an expanding invariant cone field and in this section we will show how this gives us unstable manifolds at each point of the limit set.

Instead of trying to appeal to the stable and unstable manifold theorem for dominated splittings to get local unstable manifolds we directly construct global unstable manifolds by using all the information we have about the renormalization operator and its derivative. This is done by defining a graph transform and showing

<sup>13</sup>By a *square* we mean any set diffeomorphic to the unit square.

<sup>14</sup>Note that  $\mathcal{L}$  can be embedded in  $\bar{\mathcal{L}}$  by sending  $\phi$  and  $\psi$  to singleton decompositions, which is how we can apply the propositions from Section 10 even though they are stated for decomposed Lorenz maps.



that it contracts some suitable metric similarly to the Hadamard proof of the stable and unstable manifold theorem. We are only able to show that the resulting graphs are  $\mathcal{C}^1$  since we do not have hyperbolicity. Our proof is an adaptation of the proof of Theorem 6.2.8 in Katok and Hasselblatt [1995].

**Definition 12.1.** Let  $\mathcal{A}_\Omega$  be as in Definition 8.9 and define the limit set of renormalization for types in  $\Omega$  by

$$\Lambda_\Omega = \mathcal{A}_\Omega \cap \bar{\mathcal{L}}_{\Omega^\mathbb{N}}.$$

*Remark 12.2.* Here  $\bar{\mathcal{L}}_{\Omega^\mathbb{N}}$  denotes the set of infinitely renormalizable maps with combinatorial type in  $\Omega^\mathbb{N}$  and  $\mathcal{A}_\Omega$  can intuitively be thought of as the *attractor* for  $\mathcal{R}$ . The set  $\Omega$  is the same as in Section 4, as always.

Note that by Proposition 8.11

$$\Lambda_\Omega \subset [0, 1]^2 \times (0, 1) \times \bar{\mathcal{Q}}^2,$$

where  $\bar{\mathcal{Q}}$  denotes the set of pure decompositions, see Definition 7.14.

**Theorem 12.3.** *For every  $\bar{f} = (u, v, c, \bar{\phi}, \bar{\psi}) \in \Lambda_\Omega$  there exists a unique global unstable manifold  $\mathcal{W}^u(\bar{f})$ . The unstable manifold is a graph*

$$\mathcal{W}^u(\bar{f}) = \{(\xi, \sigma(\xi)) \mid \xi \in I\},$$

where  $\sigma : I \rightarrow (0, 1) \times \bar{\mathcal{Q}}^2$  is  $\kappa$ -Lipschitz for some  $\kappa \ll 1$  (not depending on  $\bar{f}$ ). The domain  $I \subset \mathbb{R}^2$  is essentially given by

$$\pi(\mathcal{R}(\bar{\mathcal{L}}_\omega) \cap ([0, 1]^2 \times \{c\} \times \{\bar{\phi}\} \times \{\bar{\psi}\})),$$

where  $\pi$  is the projection onto the  $(u, v)$ -plane, and  $\omega$  is defined by  $\bar{f}$  being in the image  $\mathcal{R}(\bar{\mathcal{L}}_\omega)$ . Additionally,  $\mathcal{W}^u$  is  $\mathcal{C}^1$ .

*Remark 12.4.* Note that in stark contrast to the situation in the ‘regular’ stable and unstable manifold theorem we get *global* unstable manifolds which are graphs and that these are almost completely *straight* due to the Lipschitz constant being very small. The statement about the domain  $I$  is basically that  $I$  is “as large as possible.” This will be elaborated on in the proof.

Another thing to note is that we cannot say anything about the uniqueness of  $\bar{f} \in \Lambda_\Omega$  for a given combinatorics. That is, given

$$\bar{\omega} = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$$

we cannot prove that there exists a unique  $\bar{f} \in \Lambda_\Omega$  realizing this combinatorics. Instead we see a foliation of the set of maps with type  $\bar{\omega}$  by unstable manifolds. If we had a hyperbolic structure on  $\Lambda_\Omega$  this problem would go away.

**Theorem 12.5.** *Let  $\bar{f} \in \Lambda_\Omega$  and let  $\bar{\omega} \in \Omega^\mathbb{N}$ . Then  $\mathcal{W}^u(\bar{f})$  intersects the set of infinitely renormalizable maps of combinatorial type  $\bar{\omega}$  in a unique point, and the union of all such points over  $\bar{\omega} \in \Omega^\mathbb{N}$  is a Cantor set.*

*Proof.* Theorem 12.3 shows that the unstable manifolds are straight (see the above remark) and hence Lemma 12.6 enables us to apply the same arguments as in Theorem 11.4.  $\square$

**Lemma 12.6.** *There exists  $\kappa$  close to 1 such that if  $\gamma : [0, 1]^2 \rightarrow (0, 1) \times \bar{\mathcal{Q}}^2$  is  $\kappa$ -Lipschitz and  $\text{graph } \gamma \subset \bar{\mathcal{K}}$ , then  $\bar{\mathcal{L}}_\omega \cap \text{graph } \gamma$  is diffeomorphic to a square, for every  $\omega \in \Omega$ .*

*Proof.* By Theorem 11.2 the set  $\bar{\mathcal{L}}_\omega \cap \bar{\mathcal{K}}$  is a tube for every  $\omega \in \Omega$  (by *tube* we mean that the set is diffeomorphic to  $[0, 1]^2 \times X$  for some set  $X$ ). Take a tangent vector at a point in  $\partial(\mathcal{R}\bar{\mathcal{L}}_\omega) \cap \bar{\mathcal{K}}$ . Such a tangent will lie in the complement of a cone  $H_\kappa = \{\|y\| \leq \kappa\|x\|\}$  for  $\kappa < 1$  close to 1, since the projection of the image of a tube to the  $(u, v, c)$ -subspace will look like a slightly deformed cut-off part of the wedge in Figure 8 and the maximum angle of a tangent vector in the boundary of the wedge is exactly 1. By Proposition 10.1,  $D\mathcal{R}^{-1}$  maps the complement of  $H_\kappa$  into itself and hence every tube “lies in the complement of  $H_\kappa$ ”. That is, a tangent vector at a point in the boundary of a tube lies in the complement of  $H_\kappa$ , so the tubes cut the  $(u, v)$ -plane at an angle which is smaller than  $1/\kappa$ .

Now if we choose  $\kappa$  as above, then the graph of  $\gamma$  will also intersect every tube on an angle. Hence the intersection is diffeomorphic to a square. The main point here is that with  $\kappa$  chosen properly,  $\gamma$  cannot ‘fold over’ a tube and in such a way create an intersection which is not simply connected.  $\square$

*Proof of Theorem 12.3.* The proof is divided into three steps: (1) definition of the graph transform  $\Gamma$ , (2) showing that  $\Gamma$  is a contraction, (3) proof of  $\mathcal{C}^1$ -smoothness of the unstable manifold.

*Step 1.* From Lemma 4.7 we know that the parameters  $u$  and  $v$  for any map in  $\bar{\mathcal{K}}$  are uniformly close to 1 so there exists  $\mu \ll 1$  such that if we define the ‘block’

$$\bar{\mathcal{B}} = [1 - \mu, 1]^2 \times (0, 1) \times \bar{\mathcal{Q}}^2 \cap \bar{\mathcal{K}},$$

then  $\bar{\mathcal{L}}_\Omega \cap \bar{\mathcal{K}} \subset \bar{\mathcal{B}}$ ,  $1 - \mu > \phi^{-1}(c)$  and  $\mu > \psi^{-1}(c)$  for all  $(u, v, c, \phi, \psi) \in O(\bar{\mathcal{B}})$ . In other words, the block  $\bar{\mathcal{B}}$  is defined so that it contains all maps in  $\bar{\mathcal{K}}$  which are renormalizable of type in  $\Omega$  and the square  $[1 - \mu, 1]^2$  is contained in the projection of the image  $\mathcal{R}(\bar{\mathcal{L}}_\Omega \cap \bar{\mathcal{K}})$  onto the  $(u, v)$ -plane.

Fix  $\bar{f}_0 \in \Lambda_\Omega$  and  $\kappa \in (\kappa^-, 1)$ , where  $\kappa^-$  is the supremum of  $\kappa^-(\bar{f})$  defined in Proposition 10.1 and  $\kappa$  is small enough so that Lemma 12.6 applies. Associated with  $\bar{f}_0$  are two bi-infinite sequences  $\{\omega_i\}_{i \in \mathbb{Z}}$  and  $\{\bar{f}_i\}_{i \in \mathbb{Z}}$  such that  $\mathcal{R}_{\omega_i} \bar{f}_i = \bar{f}_{i+1}$  for all  $i \in \mathbb{Z}$ . Now define  $\mathcal{G}_i$ , the “unstable graphs centered on  $\bar{f}_i$ ,” as the set of  $\kappa$ -Lipschitz maps  $\gamma_i : [1 - \mu, 1]^2 \rightarrow (0, 1) \times \bar{\mathcal{Q}}^2$  such that  $\text{graph } \gamma_i \subset \bar{\mathcal{B}}$  and  $\gamma_i(\xi_i) = (c_i, \bar{\phi}_i, \bar{\psi}_i)$ , where  $\bar{f}_i = (\xi_i, c_i, \bar{\phi}_i, \bar{\psi}_i)$ .

Let  $\mathcal{G} = \prod_i \mathcal{G}_i$ . We will now define a metric on  $\mathcal{G}$ . Let

$$d_i(\gamma_i, \theta_i) = \sup_{\xi \in [1 - \mu, 1]^2} \frac{|\gamma_i(\xi) - \theta_i(\xi)|}{|\xi - \xi_i|}, \quad \gamma_i, \theta_i \in \mathcal{G}_i,$$

and define

$$d(\gamma, \theta) = \sup_{i \in \mathbb{Z}} d_i(\gamma_i, \theta_i), \quad \gamma, \theta \in \mathcal{G}.$$

This metric turns  $(\mathcal{G}, d)$  into a complete metric space. Note that it is not enough to simply use a  $\mathcal{C}^0$ -metric since we do not have a contracting subspace of  $D\mathcal{R}$ . The denominator in the definition of  $d_i$  is thus necessary to turn the graph transform into a contraction.

We can now define the graph transform  $\Gamma : \mathcal{G} \rightarrow \mathcal{G}$  for  $\bar{f}_0$ . Let  $\gamma_i \in \mathcal{G}_i$  and define  $\Gamma_i(\gamma_i)$  to be the  $\gamma'_{i+1} \in \mathcal{G}_{i+1}$  such that

$$\text{graph } \gamma'_{i+1} = \mathcal{R}_{\omega_i}(\text{graph } \gamma_i \cap \bar{\mathcal{L}}_{\omega_i}) \cap \bar{\mathcal{B}}.$$

Let us discuss why this is a well-defined map  $\Gamma_i : \mathcal{G}_i \rightarrow \mathcal{G}_{i+1}$ . Lemma 12.6 shows that  $\mathcal{R}_{\omega_i}(\text{graph } \gamma_i \cap \bar{\mathcal{L}}_{\omega_i})$  is the graph of some map  $I \subset \mathbb{R}^2 \rightarrow (0, 1) \times \bar{\mathcal{Q}}^2$ , where  $I$

is simply connected. That  $I \supset [1 - \mu, 1]^2$  is a consequence of how  $\bar{\mathcal{B}}$  was chosen. Finally, this map is  $\kappa$ -Lipschitz by Proposition 10.1.

Actually, we have cheated a little bit here since Proposition 10.1 is stated for maps satisfying the extra condition

$$1 - c_1^+(\mathcal{R}f) \geq \lambda,$$

for some  $\lambda \in (0, 1)$  not depending on  $\bar{f} \in \Lambda_\Omega$ . In defining the graph transform we should intersect  $\bar{\mathcal{L}}_{\omega_i}$  with the set defined by this condition before mapping it forward by  $\mathcal{R}_{\omega_i}$ . Otherwise we do not have enough information to deduce that the entire image is  $\kappa$ -Lipschitz as well. However, this problem is artificial. We are free to choose the constant  $\lambda$  as close to 0 as we like and we would still get the invariant cone field (although  $b_0$  may need to be increased). All this means is that domain  $I$  of the theorem is slightly smaller than it should be (we have to cut out a small part of the graph where  $v$  is very close to 0 but  $v$  is still allowed to range all the way up to 1 so this amounts to a very small part of the domain). This is one reason why we say that “ $I$  is essentially given by ...” in the statement of the theorem. The other reason is that the intersection with  $\mathcal{R}(\bar{\mathcal{L}}_\omega)$  should be taken with a surface with a small angle and not a surface which is parallel to the  $(u, v)$ -plane.

The graph transform is now defined by

$$\Gamma(\gamma) = \{\Gamma_i(\gamma_i)\}_{i \in \mathbb{Z}}, \quad \gamma = \{\gamma_i\}_{i \in \mathbb{Z}} \in \mathcal{G}.$$

We claim that  $\Gamma$  is a contraction on  $(\mathcal{G}, d)$  and hence the contraction mapping theorem implies that  $\Gamma$  has a unique fixed point  $\gamma^* \in \mathcal{G}$ . The global unstable manifolds along  $\{\bar{f}_i\}$  are then given by

$$\mathcal{W}^u(\bar{f}_{i+1}) = \text{graph } \Gamma_i(\gamma_i^*), \quad \forall i \in \mathbb{Z}.$$

In particular, this proves existence and uniqueness of the global unstable manifold at  $f_0$ . That these are the *global* unstable manifolds is a consequence of  $\bar{\mathcal{L}}_\Omega \cap \bar{\mathcal{K}} \subset \bar{\mathcal{B}}$ . Furthermore, the Lipschitz constant for these graphs is much smaller than 1 since we can pick  $\kappa$  close to  $\kappa^-$ . Again, we are cheating a little bit here since we have to cut out a small part of the domain of the graph as discussed above.

*Step 2.* We now prove that  $\Gamma$  is a contraction. The focus will be on  $\Gamma_i$  for now and to avoid clutter we will drop subscripts on elements of  $\mathcal{G}_i$  and  $\mathcal{G}_{i+1}$ . Pick  $\gamma, \theta \in \mathcal{G}_i$  and let  $\gamma' = \Gamma_i(\gamma)$  and  $\theta' = \Gamma_i(\theta)$ . Note that  $\gamma', \theta' \in \mathcal{G}_{i+1}$ .

We write

$$\mathcal{R}\bar{f} = (A(\xi, \eta), B(\xi, \eta)),$$

where  $\bar{f} = (\xi, \eta)$ ,  $\xi \in \mathbb{R}^2$  and  $A(\xi, \eta) \in \mathbb{R}^2$ . Let  $A_\gamma(\xi) = A(\xi, \gamma(\xi))$  and similarly  $B_\gamma(\xi) = B(\xi, \gamma(\xi))$ . With this notation the action of  $\Gamma_i$  is given by

$$(\xi, \gamma(\xi)) \mapsto (A_\gamma(\xi), B_\gamma(\xi)) = (\xi', \gamma'(\xi')).$$

Hence

$$d_{i+1}(\gamma', \theta') = \sup_{\xi'} \frac{\|\gamma'(\xi') - \theta'(\xi')\|}{\|\xi' - \xi_{i+1}\|} = \sup_{A_\gamma(\xi)} \frac{\|\gamma' \circ A_\gamma(\xi) - \theta' \circ A_\gamma(\xi)\|}{\|A_\gamma(\xi) - A_\gamma(\xi_i)\|}.$$

Recall that the notation here is  $(\xi_i, \gamma(\xi_i)) = \bar{f}_i$  and  $(\xi_{i+1}, \gamma'(\xi_{i+1})) = \bar{f}_{i+1}$ .

The last numerator can be estimated by

$$\begin{aligned}
& \|\gamma' \circ A_\gamma(\xi) - \theta' \circ A_\gamma(\xi)\| \\
& \leq \|\gamma' \circ A_\gamma(\xi) - \theta' \circ A_\theta(\xi)\| + \|\theta' \circ A_\gamma(\xi) - \theta' \circ A_\theta(\xi)\| \\
& \leq \|B_\gamma(\xi) - B_\theta(\xi)\| + \kappa \|A_\gamma(\xi) - A_\theta(\xi)\| \\
& \leq (\|M_4\| + \kappa \|M_2\|) \|\gamma(\xi) - \theta(\xi)\|.
\end{aligned}$$

The denominator can be bounded by Proposition 10.3

$$\|A_\gamma(\xi) - A_\gamma(\xi_i)\| \geq k \cdot \min\{|U|^{-1}, |V|^{-1}\} \cdot \|\xi - \xi_i\|.$$

Thus

$$d_{i+1}(\gamma', \theta') \leq \frac{(\|M_4\| + \kappa \|M_2\|)}{k \cdot \min\{|U|^{-1}, |V|^{-1}\}} d_i(\gamma, \theta) = \nu d_i(\gamma, \theta).$$

Theorem 9.2 shows that  $\nu \ll 1$  uniformly in the index  $i$ . Hence  $\Gamma$  is a (very strong) contraction.

*Step 3.* Going from Lipschitz to  $C^1$  smoothness of the unstable manifold is a standard argument. See for example Katok and Hasselblatt [1995, Chapter 6.2].  $\square$

#### APPENDIX A. A FIXED POINT THEOREM

The following theorem is an adaptation of Granas and Dugundji [2003, Theorem 4.7].

**Theorem A.1.** *Let  $X \subset Y$  where  $X$  is closed and  $Y$  is a normal topological space. If  $f : X \rightarrow Y$  is homotopic to a map  $g : X \rightarrow Y$  with the property that every extension of  $g|_{\partial X}$  to  $X$  has a fixed point in  $X$ , and if the homotopy  $h_t$  has no fixed point on  $\partial X$  for every  $t \in [0, 1]$ , then  $f$  has a fixed point in  $X$ .*

*Remark A.2.* Note that the statement is such that  $X$  must have nonempty interior. This follows from the assumption that  $g$  has a fixed point (since it is an extension of  $g|_{\partial X}$ ) but the requirement on the homotopy implies that  $g$  has no fixed point on  $\partial X$ .

*Proof.* Let  $F_t$  be the set of fixed points of  $h_t$  and let  $F = \bigcup F_t$ . Since  $g$  must have a fixed point  $F$  is nonempty. Since  $h_t$  has no fixed points on  $\partial X$  for every  $t$ ,  $F$  and  $\partial X$  are disjoint.

We claim that  $F$  is closed. To see this, let  $\{x_n \in F\}$  be a convergent sequence, let  $x = \lim x_n$ . Note that  $x \in X$  since  $F \subset X$  and  $X$  is closed. By definition there exists  $t_n \in [0, 1]$  such that  $x_n = h(x_n, t_n)$ . Pick a convergent subsequence  $t_{n_k} \rightarrow t$ . Since  $x_n$  is convergent  $h(x_{n_k}, t_{n_k}) = x_{n_k} \rightarrow x$ , but at the same time  $h(x_{n_k}, t_{n_k}) \rightarrow h(x, t)$  since  $h$  is continuous. Hence  $h(x, t) = x$ , that is  $x \in F$  which proves the claim.

Since  $Y$  is normal and  $\partial X$  and  $F$  are disjoint closed sets there exists a map  $\lambda : X \rightarrow [0, 1]$  such that  $\lambda|_F = 0$  and  $\lambda|_{\partial X} = 1$ . Define  $\bar{g}(x) = h(x, \lambda(x))$ . Then  $\bar{g}$  is an extension of  $g|_{\partial X}$  since if  $x \in \partial X$ , then  $\bar{g}(x) = h(x, 1) = g(x)$ . Hence  $\bar{g}$  has a fixed point  $p \in X$ . However,  $p$  must also be a fixed point of  $f$  since  $p = \bar{g}(p) = h(p, \lambda(p))$  so that  $p \in F$  and consequently  $p = \bar{g}(p) = h(p, 0) = f(p)$ .  $\square$

## APPENDIX B. THE NONLINEARITY OPERATOR

In this appendix we collect some results on the nonlinearity operator. The proofs are either simple calculations or can be found for example in the appendix of [Martens, 1998].

**Definition B.1.** Let  $\mathcal{C}^k(A; B)$  denote the set of  $k$  times continuously differentiable maps  $f : A \rightarrow B$  and let  $\mathcal{D}^k(A; B) \subset \mathcal{C}^k(A; B)$  denote the subset of orientation-preserving homeomorphisms whose inverse lie in  $\mathcal{C}^k(B; A)$ .

As a notational convenience we write  $\mathcal{C}^k(A)$  instead of  $\mathcal{C}^k(A; A)$ , and  $\mathcal{C}^k$  instead of  $\mathcal{C}^k(A; B)$  if there is no need to specify  $A$  and  $B$  (and similarly for  $\mathcal{D}^k$ ).

**Definition B.2.** The nonlinearity operator  $N : \mathcal{D}^2(A; B) \rightarrow \mathcal{C}^0(A; \mathbb{R})$  is defined by

$$(67) \quad N\phi = D \log D\phi.$$

We say that  $N\phi$  is the nonlinearity of  $\phi$ .

*Remark B.3.* Note that

$$N\phi = \frac{D^2\phi}{D\phi}.$$

**Definition B.4.** The distortion of  $\phi \in \mathcal{D}^1(A; B)$  is defined by

$$\text{Dist } \phi = \sup_{x, y \in A} \log \frac{D\phi(x)}{D\phi(y)}.$$

*Remark B.5.* We think of the nonlinearity of  $\phi \in \mathcal{D}^2(A; B)$  as the density for the distortion of  $\phi$ . To understand this remark, let  $d\mu = N\phi(t)dt$ . Assuming  $N\phi$  is a positive function, then  $\mu$  is a measure and

$$\text{Dist } \phi = \int_A d\mu,$$

since by (67)

$$\int_x^y N\phi(t)dt = \log \frac{D\phi(y)}{D\phi(x)}.$$

If  $N\phi$  is negative, then  $-N\phi(t)$  is a density. The only problem with the interpretation of  $N\phi$  as a density occurs when it changes sign. Intuitively speaking, we can still think of the nonlinearity as a *local* density of the distortion (away from the zeros of  $N\phi$ ).

Note that  $N\phi$  does not change sign in the important special case of  $\phi$  being a pure map (i.e. a restriction of  $x^\alpha$ ). So the (absolute value of the) nonlinearity is the density for the distortion of pure maps.

**Lemma B.6.** *The kernel of  $N : \mathcal{D}^2(A; B) \rightarrow \mathcal{C}^0(A; \mathbb{R})$  equals the orientation-preserving affine map that takes  $A$  onto  $B$ .*

**Lemma B.7.** *The nonlinearity operator  $N : \mathcal{D}^2(A; B) \rightarrow \mathcal{C}^0(A; \mathbb{R})$  is a bijection. In the specific case of  $A = B = [0, 1]$  the inverse is given by*

$$(68) \quad N^{-1}f(x) = \frac{\int_0^x \exp\{\int_0^s f(t)dt\}ds}{\int_0^1 \exp\{\int_0^s f(t)dt\}ds}.$$

**Lemma B.8** (The chain rule for the nonlinearity operator). *If  $\phi, \psi \in \mathcal{D}^2$  then*

$$(69) \quad N(\psi \circ \phi) = N\psi \circ \phi \cdot D\phi + N\phi.$$

**Definition B.9.** We turn  $\mathcal{D}^2(A; B)$  into a Banach space by inducing the usual linear structure and uniform norm of  $\mathcal{C}^0(A; \mathbb{R})$  via the nonlinearity operator. That is, we define

$$(70) \quad \alpha\phi + \beta\psi = N^{-1}(\alpha N\phi + \beta N\psi),$$

$$(71) \quad \|\phi\| = \sup_{t \in A} |N\phi(t)|,$$

for  $\phi, \psi \in \mathcal{D}^2(A; B)$  and  $\alpha, \beta \in \mathbb{R}$ .

**Lemma B.10.** *If  $\phi \in \mathcal{D}^2(A; B)$  then*

$$(72) \quad e^{-|y-x| \cdot \|\phi\|} \leq \frac{D\phi(y)}{D\phi(x)} \leq e^{|y-x| \cdot \|\phi\|},$$

$$(73) \quad \frac{|B|}{|A|} \cdot e^{-\|\phi\|} \leq D\phi(x) \leq \frac{|B|}{|A|} \cdot e^{\|\phi\|},$$

$$(74) \quad |D^2\phi(x)| \leq \frac{|B|}{|A|} \cdot \|\phi\| \cdot e^{\|\phi\|},$$

for all  $x, y \in A$ .

**Lemma B.11.** *If  $\phi, \psi \in \mathcal{D}^2(A; B)$  then*

$$(75) \quad |\phi(x) - \psi(x)| \leq (e^{2\|\phi-\psi\|} - 1) \cdot \min\{\phi(x), 1 - \phi(x)\},$$

$$(76) \quad e^{-\|\phi-\psi\|} \leq \frac{D\phi(x)}{D\psi(x)} \leq e^{\|\phi-\psi\|},$$

for all  $x \in A$ .

**Definition B.12.** Let  $\zeta_J : [0, 1] \rightarrow J$  be the affine orientation-preserving map taking  $[0, 1]$  onto an interval  $J$ .

Define the zoom operator  $Z : \mathcal{D}^2(A; B) \rightarrow \mathcal{D}^2([0, 1])$  by

$$(77) \quad Z\phi = \zeta_B^{-1} \circ \phi \circ \zeta_A.$$

*Remark B.13.* Note that if  $\phi \in \mathcal{D}(A; B)$ , then  $B = \phi(A)$  so  $Z\phi$  only depends on  $\phi$  and  $A$  (not on  $B$ ). We will often write  $Z(\phi; A)$  instead of  $Z\phi$  in order to emphasize the dependence on  $A$ .

**Lemma B.14.** *If  $\phi \in \mathcal{D}^2(A; B)$  then*

$$(78) \quad Z(\phi^{-1}) = (Z\phi)^{-1},$$

$$(79) \quad N(Z\phi) = |A| \cdot N\phi \circ \zeta_A,$$

$$(80) \quad \|Z\phi\| = |A| \cdot \|\phi\|.$$

## APPENDIX C. THE SCHWARZIAN DERIVATIVE

In this appendix we collect some results on the Schwarzian derivative. Proofs can be found in de Melo and van Strien [1993, Chapter IV].

**Definition C.1.** The Schwarzian derivative  $S : \mathcal{D}^3(A; B) \rightarrow \mathcal{C}^0(A; \mathbb{R})$  is defined by

$$(81) \quad Sf = D(Nf) - \frac{1}{2}(Nf)^2.$$

*Remark C.2.* Note that

$$Sf = \frac{D^3f}{Df} - \frac{3}{2} \left[ \frac{D^2f}{Df} \right]^2.$$

**Lemma C.3** (The chain rule for the Schwarzian derivative). *If  $f, g \in \mathcal{D}^3$ , then*

$$(82) \quad S(f \circ g) = Sf \circ g \cdot (Dg)^2 + Sg.$$

**Lemma C.4** (Koebe Lemma). *If  $f \in \mathcal{D}^3((a, b); \mathbb{R})$  and  $Sf \geq 0$ , then*

$$(83) \quad |Nf(x)| \leq 2 \cdot [\min\{|x - a|, |x - b|\}]^{-1}.$$

**Corollary C.5.** *Let  $\tau > 0$  and let  $f \in \mathcal{D}^3(A; B)$ . If  $f$  extends to a map  $F \in \mathcal{D}^3(I; J)$  with  $SF < 0$  and if  $J \setminus B$  has two components, each having length at least  $\tau|B|$ , then*

$$\|Zf\| \leq e^{2/\tau} \cdot 2/\tau.$$

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